From the Generalized Bruns Transformation to Variations of the Solution of the Geodetic Boundary Value Problem

Reiner Rummel and Martin van Gelderen

1 Introduction

Erik Grafarend has inspired geodesy in so many ways, that selecting one area automatically implies to do injustice to all others. Nevertheless, one can probably say that the main focus of his research is mathematical, physical and statistical geodesy. But this is also to say his focus was and is geodesy as a whole. There is a deep love for geometry in Grafarend's work. This is also reflected in his contributions to physical geodesy. Periods where he expressed his thoughts in plane coordinates were followed by others in which he used spherical or ellipsoidal coordinates. For us 'Earthlings' it is impossible to follow his pace. All what we can try is to select one single aspect of his enormous work and reflect on it.

This shall be tried in the sequel, where we take as point of departure his publication *The Bruns transformation and a dual setup of geodetic observational equations*. It was published in 1980 in Washington, D.C., during his stay at the National Geodetic Survey of the US. It is well known that observables in physical geodesy depend on both the gravity field and the position where the measurement is taken. When establishing linear boundary conditions the position part can be eliminated by appropriate combination of observations; e.g. of potential and gravity anomaly. It is the well-known Bruns transformation. In the above work Grafarend could show that the Bruns transformation can be generalized to three or more dimensions and to various observables of physical geodesy. It also shows that the principle of elimination of unknowns, known from classical adjustment theory, can be translated to field quantities, too, and in particular to boundary functions as they are met when solving the geodetic boundary value problem (GBVP). This also means that free boundary value problems can be transformed into a fixed form to which standard solutions apply. For us this was the starting point for the solution of the GBVP in a more generalized fashion, that includes uniquely determined as well as overdetermined cases. We refer to Rummel and Teunissen (1986), Rummel *et al.* (1989), Rummel and Van Gelderen (1992) and Rummel and Van Gelderen (1999).

In this article, dedicated to Erik Grafarend at the occasion of his 60th birthday, we deal with GBVPs in several coordinate systems, their solution by separation and their determination. The determination step is the procedure of fixing the unknown parameters of the mathematical solution of the Laplace equation. It is achieved on the basis of gravimetric observations (potential differences, gravity anomalies, deflections of the vertical, torsion balance measurements etc.) carried out at the Earth's surface or reduced to some reference surface. The generalized Bruns transformation is thereby employed to arrive at boundary conditions free of geometric unknowns, e.g. coordinate or height corrections. Now, in 'GPS-age', one may argue that the geometry part of GBVPs is anyway taken care of by 3D positioning. In reality we still have to deal mostly with old data collected over decades. Our classical GBVPs will therefore still maintain their relevance for quite some time.

2 Solution of the Laplace equation by separation

Moon and Spencer (1961) have shown the solution of the three-dimensional Laplace equation for eleven orthogonal coordinate systems. For them a relatively simple solution by separation is possible. The coordinate system is usually chosen so as to fit best possible to the surface on which the boundary data are given. In geodesy spherical, rectangular, circular-cylinder, oblate spheroidal and ellipsoidal coordinates are of relevance. We shall deal only with the spherical, rectangular and circular-cylinder cases here and derive some special cases from them that may be of interest. The procedure consists of the following four steps (ibid.):

1. Formulation of Laplace equation in the chosen coordinate system.
2. Separation into a set of three second-order ordinary differential equations.
3. Solution of the ordinary differential equations.
4. Determination of the mathematical solution, as obtained from the superposition of the solutions under 3, by a set of approximate boundary conditions. Formulation of a closed solution whenever possible.

The first three steps have been solved by Moon and Spencer (1961) once for all. Thus this part can be kept rather short. We start with an outline of the general approach.
In the coordinate system $\{x^a\} = \{x^1, x^2, x^3\}$ Laplace equation reads

$$\nabla^2 V = \nabla_{ab} V g^{ab} = 0, \tag{1}$$

where ∇_{ab} is the second covariant derivative operator and g^{ab} the metric tensor of the coordinate frame. For orthogonal curvilinear coordinates $g^{ab} = 0 \quad (a \neq b)$ and Laplace equation can be written as (Moon and Spencer, 1961, eq. 1.09)

$$\nabla^2 V = \frac{1}{\sqrt{g}} \sum_{a=1}^{3} \nabla_a \left(\frac{\sqrt{g}}{g_{aa}} \nabla_a V \right) = 0.$$

If the coordinate system fulfills certain conditions (ibid.) the solution of Laplace's equation can be found by the *separation of variables*. For V we substitute thereby

$$V(x^1, x^2, x^3) = f(x^1) g(x^2) h(x^3)$$

into (1) and three independent, second-order differential equations are obtained. Their general solution is written as

$$f_{\ell m}(x^1), \quad g_{\ell m}(x^2) \quad \text{and} \quad h_{\ell m}(x^3),$$

respectively, where ℓ and m are two constants of integration. Their possible values will be determined by the type of solution required. The general solution of Laplace equation is obtained from a linear combination of all possible solutions. Assuming for simplicity that ℓ, m take only integer values we can write

$$V(x^1, x^2, x^3) = \sum_{\ell m} a_{\ell m} f_{\ell m}(x^1) g_{\ell m}(x^2) h_{\ell m}(x^3).$$

The value of the constants $a_{\ell m}$ has to be determined from boundary conditions on some surface S. Generally this is not an easy task but for some special cases the solution can be found easily. If the geometry of S coincides with one of the coordinate surfaces, e.g., $x^3 =$ constant, and if the functions $f_{\ell m}(x^1) g_{\ell m}(x^2)$ form a complete basis in some function space on S then the given boundary function $b(x^1, x^2)$ on S can be written as

$$b(x^1, x^2) = \sum_{\ell m} b_{\ell m} f_{\ell m}(x^1) g_{\ell m}(x^2).$$

With the Dirichlet boundary condition

$$V(x^1, x^2, x^3 = \text{constant}) = b(x^1, x^2)$$

the solution of the boundary value problem is

$$a_{\ell m} = \frac{b_{\ell m}}{h_{\ell m}(x^3 = \text{constant})} \Rightarrow$$

$$V(x^1, x^2, x^3) = \sum_{\ell m} \frac{b_{\ell m}}{h_{\ell m}(x^3 = \text{constant})} f_{\ell m}(x^1) g_{\ell m}(x^2) h_{\ell, m}(x^3).$$

For boundary conditions of other type an analogous solution can be found.

2.1 Solution in spherical coordinates

In spherical coordinates $\{\theta, \lambda, r\}$ = {co-latitude, longitude, radial distance}, Laplace equation applied to a potential field V takes the well-known form

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial \lambda^2} = 0. \tag{2}$$

After insertion of first

$$V(\theta, \lambda, r) = Y(\theta, \lambda) h(r) \tag{3a}$$

and then

$$Y(\theta, \lambda) = f(\theta) g(\lambda) \tag{3b}$$

it can be separated, with (3a), into

$$\frac{\partial^2 Y}{\partial \theta^2} + \cot\theta \frac{\partial Y}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial V}{\partial \lambda} + n(n+1)Y = 0 \tag{4}$$

and

$$\frac{\partial^2 h}{\partial r^2} + \frac{2}{r}\frac{\partial h}{\partial r} - \frac{n(n+1)}{r^2} h = 0 \tag{5}$$

($n \in \mathbb{N}$). Eq. (4) is the so-called surface Laplace or Beltrami equation with $Y(\theta, \lambda)$ the surface spherical harmonics. After insertion of (3b) one arrives at ($m \in \mathbb{Z}, |m| \leq n$)

$$\frac{\partial^2 f}{\partial \theta^2} + \cot\theta \frac{\partial f}{\partial \theta} + \left[n(n+1) - \frac{m^2}{\sin^2\theta} \right] f = 0, \tag{6}$$

and

$$\frac{\partial^2 g}{\partial \lambda^2} + m^2 g = 0. \tag{7}$$

The solutions of the ordinary second order differential equations (5),(6) and (7) are:

$$h(r) = a r^n + b r^{-(n+1)} \tag{8a}$$

$$f(\theta) = a P_{nm}(\cos\theta) + b\, Q_{nm}(\cos\theta) \tag{8b}$$

with $P_{nm}(\cos\theta)$ and $Q_{nm}(\cos\theta)$ the associated Legendre functions of the first and second kind, respectively, and

$$g(\lambda) = a \sin m\lambda + b \cos m\lambda \tag{9a}$$

$$= c \exp(im\lambda) + d \exp(-im\lambda). \tag{9b}$$

The two solutions (9a,9b) are equivalent. We select (9b) because it leads to a more compact form.With the surface spherical harmonics

$$Y_{nm}(\theta,\lambda) = P_{nm}(cos\theta)\exp(im\lambda) \tag{10}$$

and equivalently

$$Z_{nm}(\theta,\lambda) = Q_{nm}(cos\theta)\exp(im\lambda) \tag{11}$$

the complete set of solutions for all admissible integer degrees n and orders m becomes:

$$\begin{aligned} V(\theta,\lambda,r) = &\sum_{n=0}^{\infty} r^{-(n+1)} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\theta,\lambda) \\ &+ \sum_{n=0}^{\infty} r^{n} \sum_{m=-n}^{n} b_{nm} Y_{nm}(\theta,\lambda) \\ &+ \sum_{n=0}^{\infty} r^{-(n+1)} \sum_{m=-n}^{n} c_{nm} Z_{nm}(\theta,\lambda) \\ &+ \sum_{n=0}^{\infty} r^{n} \sum_{m=-n}^{n} d_{nm} Z_{nm}(\theta,\lambda). \end{aligned} \tag{12}$$

Special case − V independent of λ (axial-symmetric) For V independent of λ eq. (6) changes into

$$\frac{\partial^2 f}{\partial\theta^2} + \cot\theta\frac{\partial f}{\partial\theta} + n(n+1)f = 0, \tag{13}$$

the characteristic equation for the Legendre polynomials, which has the solution

$$f(\theta) = aP_n(cos\theta) + b\,Q_n(cos\theta). \tag{14}$$

For this case the complete set of solutions reads:

$$\begin{aligned} V(\theta,\lambda) = &\sum_{n=0}^{\infty} a_n r^{-(n+1)} P_n(cos\theta) + \sum_{n=0}^{\infty} b_n r^n P_n(cos\theta) \\ &+ \sum_{n=0}^{\infty} c_n r^{-(n+1)} Q_n(cos\theta) + \sum_{n=0}^{\infty} d_n r^n Q_n(cos\theta). \end{aligned} \tag{15}$$

2.2 Solution in Cartesian coordinates

The Cartesian coordinate triple $\{x, y, z\}$ is arbitrarily chosen to mean {North direction in the $\{x, y\}$-plane, East direction in the $\{x, y\}$-plane, positive up}. Laplace equation in Cartesian coordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{16}$$

is dealt with by

$$V(x, y, z) = X(x)Y(y)Z(z). \tag{17}$$

It separates either into

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0, \tag{18a}$$

$$\frac{\partial^2 Y}{\partial y^2} + \ell^2 Y = 0 \quad \text{and} \tag{18b}$$

$$\frac{\partial^2 Z}{\partial z^2} - (k^2 + \ell^2)Z = 0 \tag{18c}$$

or into

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0, \tag{19a}$$

$$\frac{\partial^2 Y}{\partial y^2} - \ell^2 Y = 0 \quad \text{and} \tag{19b}$$

$$\frac{\partial^2 Z}{\partial z^2} - (k^2 - \ell^2) Z = 0. \tag{19c}$$

In the first case the solutions of the first two ordinary second-order differential equations are:

$$X(x) = a\exp(-ikx) + b\exp(ikx) \tag{20a}$$

$$Y(y) = a\exp(-i\ell y) + b\exp(i\ell y) \tag{20b}$$

or equivalently written in sin/cos-terms. This leads for $Z(z)$ to

$$Z(z) = a\exp(-\sqrt{k^2+\ell^2}z) + b\exp(\sqrt{k^2+\ell^2}z). \tag{20c}$$

In the second case we obtain

$$X(x) = a\exp(-ikx) + b\exp(ikx), \tag{21a}$$

$$Y(y) = a\exp(-\ell y) + b\exp(\ell y) \quad \text{and} \tag{21b}$$

$$Z(z) = a\exp(-\sqrt{k^2-\ell^2}z) + b\exp(\sqrt{k^2-\ell^2}z). \tag{21c}$$

The complete set of solutions becomes – with k and ℓ assumed to be integers!:

$$\begin{aligned} V(x,y,z) = &\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_{k\ell}\exp[i(kx+\ell y)(-\sqrt{k^2+\ell^2}z)] \\ &+ \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} b_{k\ell}\exp[i(kx+\ell y)\sqrt{k^2+\ell^2}z] \end{aligned} \tag{22}$$

or, alternatively, from eqs. (21a)-(21c):

$$\begin{aligned} V(x,y,z) = &\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_{k\ell}\exp\left[(ikx+\ell y-\sqrt{k^2-\ell^2}z\right] \\ &+ \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} b_{k\ell}\exp\left[ikx+\ell y+\sqrt{k^2-\ell^2}z\right]. \end{aligned} \tag{23}$$

Special case – V independent of y For V independent of y two cases are distinguished:

Case 1 :

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0 \tag{24a}$$

and

$$\frac{\partial^2 Z}{\partial z^2} - k^2 Z = 0 \tag{24b}$$

with the solutions

$$X(x) = a\exp(-ikx) + b\exp(ikx) \tag{25a}$$

$$Z(z) = a\exp(-kz) + b\exp(kz).$$

Case 2 :

$$\frac{\partial^2 X}{\partial x^2} = \frac{\partial^2 Z}{\partial z^2} = 0 \tag{26}$$

with the solution

$$X(x) = a + bx \quad \text{and} \tag{27a}$$
$$Z(z) = a + bz. \tag{27b}$$

For the complete solutions we find:

Case 1 :

$$V(x,z) = \sum_{k=-\infty}^{\infty} a_k \exp\left[ikx - |k|z\right] + \sum_{k=-\infty}^{\infty} b_k \exp\left[ikx + |k|z\right] \tag{28}$$

and

Case 2 :

$$V(x,z) = a + bx + cz + dxz. \tag{29}$$

2.3 Solution in circular cylinder coordinates

In circular cylinder coordinates $\{r, \lambda, z\}$ = {radial in $\{x, y\}$-plane, longitude in $\{x, y\}$-plane, positive up} Laplace equation becomes

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \lambda^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{30}$$

With

$$V(r, \lambda, z) = f(r)g(\lambda)Z(z) \tag{31}$$

it separates into

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - (k^2 + \frac{m^2}{r^2})f = 0, \tag{32a}$$
$$\frac{\partial^2 g}{\partial \lambda^2} + m^2 g = 0, \tag{32b}$$
$$\frac{\partial^2 Z}{\partial z^2} + k^2 Z = 0. \tag{32c}$$

The solutions are e.g.

$$f(r) = aI_m(kr) + bK_m(kr) \tag{33a}$$
$$g(\lambda) = a\exp(-im\lambda) + b\exp(im\lambda) \tag{33b}$$
$$Z(z) = a\exp(-ikz) + b\exp(ikz) \tag{33c}$$

with the modified Bessel functions of the first kind I_m and of the second kind K_m; see Lehner (1996). Alternatively, one could separate into

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + (k^2 - \frac{m^2}{r^2})f = 0, \tag{34}$$

$$\frac{\partial^2 g}{\partial \lambda^2} + m^2 g = 0, \quad \text{and} \tag{35}$$

$$\frac{\partial^2 Z}{\partial z^2} - k^2 Z = 0, \tag{36}$$

with solutions

$$f(r) = aJ_m(kr) + bN_m(kr) \tag{37a}$$

$$g(\lambda) = a\exp(-im\lambda) + b\exp(im\lambda) \tag{37b}$$

$$Z(z) = a\exp(-kz) + b\exp(kz) \tag{37c}$$

with the Bessel function J_m and the Neumann functions (Bessel functions of the second kind) N_m. The complete set of solutions becomes now

$$V(r,\lambda,z) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{km} \exp(i(m\lambda + kz)) K_m(kr)\, dk + \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} b_{km} \exp(i(m\lambda + kz)) I_m(kr)\, dk \tag{38}$$

or

$$V(r,\lambda,z) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{km} \exp(im\lambda + kz)) J_m(kr)\, dk + \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} b_{km} \exp(i(m\lambda + kz)) N_m(kr)\, dk. \tag{39}$$

Special case – V independent of z For $V(r,\lambda)$ the two ordinary second-order differential equations become

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{m^2}{r^2} f = 0 \tag{40}$$

and

$$\frac{\partial^2 g}{\partial \lambda^2} + m^2 g = 0 \tag{41}$$

with the well-known solutions

$$f(r) = ar^m + br^{-m} \tag{42}$$

and

$$g(\lambda) = a\exp(-im\lambda) + b\exp(im\lambda). \tag{43}$$

Special attention has to be paid to the case $m = 0$, for then it is

$$f(r) = a + b\ln r \quad \text{and} \tag{44a}$$

$$g(\lambda) = a. \tag{44b}$$

This results in the complete set of solutions

$$V(r,\lambda) = c_0 + c_1 \ln r + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} a_m r^{-|m|} \exp(im\lambda) + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} b_m r^{|m|} \exp(im\lambda). \tag{45}$$

See also (Walter, 1971).

3 Determination of the solution by boundary conditions

In the previous chapter the solution of Laplace equation was given in spherical, Cartesian and circular-cylinder coordinates. Also three special cases are included. Now the mathematical solutions are to be determined with the help of boundary conditions. Only for the spherical case this step shall be discussed explicitly. For the two other coordinate systems and for the special cases the solutions will only be summarized. Throughout only "exterior" problems will be treated with a regularity condition at infinity for V.

3.1 Determination of the solution for the exterior of a sphere with radius $r = R$

The mathematical solution is given in eq. (12). The following assumptions hold:

(α) $\mathcal{D}V(\theta, \lambda, r = R) = f(\theta, \lambda)$
The linear differential operator D applied to the potential V takes the values of the boundary function f on the sphere $S(O, r = R)$.

(β) $\lim_{r\to\infty} V(\theta, \lambda, r) = 0$
The potential V takes the values zero of the hypothetical boundary function $f'(\theta, \lambda) = 0$ on a sphere S' with $\lim r \to \infty$ (regularity condition).

Determination:

- since all $Z_{nm}(\theta, \lambda)$ take the value infinity for $\theta = 0$ (= z-axis), for V in order not to become infinity all coefficients c_{nm} and d_{nm} must be zero:

$$c_{nm} = d_{nm} = 0; \tag{46}$$

- in order to meet boundary condition (β) with $\lim_{r\to\infty} r^n = \infty$ all b_{nm} must be zero:

$$(\beta) \qquad b_{nm} = 0. \tag{47}$$

It remains to determine the coefficients a_{nm}. We discuss three cases of (α): the boundary conditions of type Dirichlet, Neumann and Stokes.

Type Dirichlet ($\mathcal{D} = 1$): Then with

$$\begin{aligned} A_{nm} =& \frac{1}{4\pi} \iint f(\theta, \lambda) Y_{nm}(\theta, \lambda)\, d\sigma \\ =& \langle f, Y_{nm}(\theta, \lambda) \rangle \end{aligned} \tag{48}$$

(we now assume that the spherical harmonics Y_{nm} are orthonormal) and therefore

$$f(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} Y_{nm}(\theta, \lambda), \tag{49}$$

it follows by comparison of coefficients (compare (12) and (49))

$$R^{-(n+1)} a_{nm} = A_{nm} \tag{50}$$

and therefore

$$a_{nm} = R^{n+1} A_{nm}. \qquad (\alpha - I) \tag{51}$$

This could be called the spectral solution of the Dirichlet problem for the exterior of the sphere. Insertion into (12) yields

$$V(\theta,\lambda,r)=\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1}\sum_{m=-n}^{n}A_{nm}Y_{nm}(\theta,\lambda). \tag{52}$$

With (48) and the addition theorem

$$(2n+1)P_n(\cos\psi)=\sum_{m=-n}^{n}Y_{nm}(\theta,\lambda)Y_{nm}(\theta',\lambda') \tag{53}$$

one arrives at

$$\begin{aligned}V(\theta,\lambda,r)&=\frac{1}{4\pi}\iint\left\{\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1}(2n+1)P_n(\cos\psi)\right\}f(\theta',\lambda')\,d\sigma'\\&=\frac{1}{4\pi}\iint D(\psi;r,R)f(\theta',\lambda')\,d\sigma'.\end{aligned} \tag{54}$$

This is the closed solution of the spherical Dirichlet problem, in geodesy referred to as *Poisson equation.*

Type Neumann ($\mathcal{D}=\frac{\partial}{\partial r}$):

$$(\alpha)\qquad \left.\frac{\partial V}{\partial r}\right|_{r=R}=f(\theta,\lambda). \tag{55}$$

Then (50) turns into

$$-(n+1)R^{-(n+2)}a_{nm}=A_{nm} \tag{56}$$

and therefore

$$a_{nm}=-\frac{R}{n+1}R^{n+1}A_{nm}.\qquad(\alpha-II) \tag{57}$$

Now insertion into (12) gives

$$V(\theta,\lambda,r)=-\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1}\frac{R}{n+1}\sum_{m=-n}^{n}A_{nm}Y_{nm}(\theta,\lambda). \tag{58}$$

Again a closed solution is possible:

$$\begin{aligned}V(\theta,\lambda,r)&=-\frac{R}{4\pi}\iint\left\{\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1}\frac{2n+1}{n+1}P_n(\cos\psi)\right\}f(\theta',\lambda')\,d\sigma'\\&=\frac{R}{4\pi}\iint H(\psi;r,R)f(\theta',\lambda')\,d\sigma'.\end{aligned} \tag{59}$$

The solution is called *Hotine integral* in geodesy (apart from the minus sign).

Type Stokes ($\mathcal{D}=-(\frac{\partial}{\partial r}+\frac{2}{r})$):

$$(\alpha)\qquad -\left(\frac{\partial V}{\partial r}+\frac{2}{r}V\right)_{r=R}=f(\theta,\lambda). \tag{60}$$

The comparison of coefficients yields

$$(n-1)R^{-(n+2)}a_{nm}=A_{nm} \tag{61}$$

Table 1: Closed expressions of integral kernels for the spherical GBVP .

$D(\psi; r, R)$	$\frac{R(r^2-R^2)}{\ell^3}$	H-M 1-89
$H(\psi; r, R)$	$-\frac{2}{\ell} + \frac{1}{R} \ln \frac{\ell+R-r\cos\psi}{r(1-\cos\psi)}$	PPV 1656
$St(\psi; r, R)$	$\frac{2R}{\ell} + \frac{R}{r} - 3\frac{R\ell}{r^2} - \frac{R^2}{r^2}\cos\psi(5 + 3\ln\frac{r-R\cos\psi+\ell}{2r})$	H-M 2-162

and therefore

$$a_{nm} = \frac{R}{n-1} R^{n+1} A_{nm} \qquad \text{for} \qquad n \neq 1. \tag{62}$$

Here a complications arises due to the singularity for $n = 1$. Thus $f(\theta, \lambda)$ has to meet an additional condition for $n = 1$:

$$A_{1m} = \frac{1}{4\pi} \iint f(\theta, \lambda) Y_{1m}(\theta, \lambda)\, d\sigma = 0, \tag{63}$$

whereas the a_{1m} remain undetermined by $f(\theta, \lambda)$; see e.g. Rummel (1995). The solution reads

$$V(\theta, \lambda, r) = A_{00} + \frac{R}{4\pi} \iint St(\psi; r, R) f(\theta', \lambda')\, d\sigma' \\ + a_{1,-1}Y_{1,-1}(\theta, \lambda) + a_{1,0}Y_{1,0}(\theta, \lambda) + a_{1,1}Y_{1,1}(\theta, \lambda). \tag{64}$$

This is the well-known *Stokes integral* formula.
Analytical expressions of D, H and St are summarized in Table 1.
The determination of special case (15) with V independent of λ is completely analogous to the previous one. Throughout it is

$$b_n = c_n = d_n = 0,$$

and we have

$$f(\theta) = \sum_n A_n P_n(\cos\theta) \Leftrightarrow A_n = \frac{2n+1}{2} \int_0^\pi f(\theta) P_n(\cos\theta) \sin\theta\, d\theta.$$

Dirichlet:

$$a_n = R^{n+1} A_n,$$

$$V(\theta, r) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} A_n P_n(\cos\theta),$$

$$V(\theta, r) = \int_0^\pi D(\theta, \theta') f(\theta') \sin\theta'\, d\theta$$

with

$$D(\theta, \theta') = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \frac{2n+1}{2} P_n(\cos\theta) P_n(\cos\theta'). \tag{65}$$

Type Neumann:

$$a_n = -\frac{R}{n+1} R^{n+1} A_n$$

$$V(\theta, r) = -\sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \frac{R}{n+1} A_n P_n(\cos\theta) \\ = \int_0^\pi N(\theta, \theta') f(\theta') \sin\theta'\, d\theta$$

with

$$N(\theta, \theta') = -R \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \frac{2n+1}{2n+2} P_n(\cos\theta) P_n(\cos\theta'). \tag{66}$$

Type Stokes:

$$a_n = \frac{R}{n-1} R^{n+1} A_n \qquad \text{for} \qquad n \neq 1,$$

$$V(\theta, r) = A_0 + \sum_{n=2}^{\infty} \frac{R}{n-1} \left(\frac{R}{r}\right)^{n+1} A_n P_n(\cos\theta) + a_1 \cos\theta$$
$$= A_0 + \int_0^{\pi} St(\theta, \theta') f(\theta') \sin\theta' \, d\theta + a_1 \cos\theta$$

with

$$St(\theta, \theta') = R \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^{n+1} \frac{2n+1}{2n-2} P_n(\cos\theta) P_n(\cos\theta'). \tag{67}$$

Closed analytical expressions for D, N and St (eqs. (65), (66) and (67)) have not been derived.

3.2 Determination of the solution in rectangular coordinates for the upper half space $z \geq z_0$

Only the solution of the Dirichlet and Neumann problem will be given in this case.

$$(\beta) \qquad \lim_{z \to \infty} V = 0 \qquad \text{regularity}$$

requires in (22)

$$b_{k\ell} = 0.$$

It is assumed that the given boundary functions are periodic (chess board pattern) with the same period T in x and y direction and write

$$f(x, y) = \sum_k \sum_\ell A_{k\ell} \exp[i(kx + \ell y)].$$

Type Dirichlet

$$a_{k\ell} = \exp(\sqrt{k^2 + \ell^2} z_0) A_{k\ell},$$

$$V(x, y, z) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} A_{k\ell} \exp[i(kx + \ell y) - \sqrt{k^2 + \ell^2}(z - z_0)],$$

$$V(x, y, z) = \frac{1}{T^2} \int_0^T \int_0^T D(x, y, z; x', y', z') f(x, y) \, dx \, dy$$

with

$$D(x, y, z; x', y') = -\frac{2(z - z_0)}{((x - x')^2 + (y - y')^2 + (z - z_0)^2)^{3/2}}.$$

Type Neumann:

$$a_{k\ell} = -\frac{1}{\sqrt{k^2 + \ell^2}} \exp(\sqrt{k^2 + \ell^2} z_0) A_{k\ell},$$

$$V(x, y, z) = -\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} \frac{A_{k\ell}}{\sqrt{k^2 + \ell^2}} \exp[i(kx + \ell y) - \sqrt{k^2 + \ell^2}(z - z_0)],$$

and (see (Kertz, 1973, Table 4)):

$$N(x, y, z; x', y') = \frac{2}{\sqrt{((x - x')^2 + (y - y')^2 + (z - z_0)^2)}}.$$

Special case: V independent of y

$$(\beta) \quad \lim_{z\to\infty} V = 0$$

requires in (28))

$$b_k = 0.$$

Type Dirichlet:

$$a_k = A_k \exp(|k|z_0)$$

$$V(x,z) = \sum_{k=-\infty}^{\infty} A_k \exp[ikx - |k|(z - z_0)]$$

and

$$D(x,z;x') = \frac{2(z - z_0)}{(x - x')^2 + (z - z_0)^2}.$$

Type Neumann:

$$a_k = -\frac{1}{k} A_k \exp(|k|z_0) \qquad \text{for} \qquad k \neq 0$$

$$V(x,z) = -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{A_k}{k} \exp[ikx - |k|(z - z_0)]$$

and

$$N(x,z;x') = \ln[(x - x')^2 + (z - z_0)^2].$$

3.3 Determination of the solution in circular cylinder coordinates for the outside of the boundary cylinder $r \geq R$

Also in this case only the solutions for Dirichlet and Neumann are given.

$$(\beta) \qquad \lim_{r\to\infty} V = 0 \qquad \text{regularity}$$

requires in (38) that

$$b_{k\ell} = 0.$$

Type Dirichlet:

$$a_{k\ell} = \frac{1}{K_m(kR)} A_{km}$$

$$V(r,\lambda,z) = \int_k \sum_m A_{km} \frac{K_m(kr)}{K_m(kR)} \exp[i(m\lambda + kz)]\, dk$$

Type Neumann:

$$\left.\frac{\partial V}{\partial r}\right|_{r=R} = \int_k \sum_m a_{km} \exp[i(m\lambda + kz)] \frac{ik}{2} (K_{m-1}(kR) - K_{m+1}(kR))\, dk;$$

see e.g. Lebedev (1965)

$$\Rightarrow a_{km} = A_{km} \frac{2}{ik(K_{m-1}(kR) - K_{m+1}(kR))}.$$

Special case: V independent of z (circle)

$$(\beta) \qquad \lim_{r\to\infty} V = 0 \qquad \text{regularity}$$

requires in (45) that

$$c_1 = 0 \quad \text{and} \quad b_m = 0.$$

Type Dirichlet:

$$a_m = R^{|m|} A_m,$$

$$V(r,\lambda) = \sum_{m=-\infty}^{\infty} \left(\frac{R}{r}\right)^{|m|} A_m \exp(im\lambda) \qquad \text{and} \quad a_0 = c_0$$

$$V(r,\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - R^2}{\ell^2} f(\lambda)\, d\lambda$$

i.e.

$$D(r,\lambda; r',\lambda') = \frac{r^2 - R^2}{\ell^2}$$

with

$$\ell^2 = r^2 + R^2 - 2rR\cos(\lambda - \lambda').$$

Type Neumann:

$$a_m = -\frac{1}{|m|} R^{-(|m|+1)} A_m \qquad (m \neq 0)$$

$$V(r,\lambda) = -\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{R}{|m|}\left(\frac{R}{r}\right)^{|m|} \exp(im\lambda)$$

and

$$N(r,\lambda; R,\lambda') = -R \ln \frac{r^2}{r^2 + R^2 - 2rR\cos(\lambda - \lambda')}.$$

4 Discussion

- For the determination of the solutions of the GBVPs in the various coordinate systems boundary conditions of the type

$$\mathcal{D}V(x^1, x^2, x^3 = const) = f(x^1, x^2)$$

  have to be available on the surface $x^3 = const$. In geodesy this condition often results from the generalized Bruns transformation, as shown in (Grafarend, 1980). The boundary surface is actually the telluroid (determined by some mapping). As the telluroid is a surface too complicated, it is approximated in practise by an ellipsoid, sphere or a tangent plane.

- The linear differential operator $\mathcal{D}$ requires often to consider boundary value problems different from the classical Dirichlet and Neumann ones. Even the Stokes boundary condition is not a classical boundary value problem of the third kind (Robin or Poincaré). Each of them requires, therefore, careful analysis of their singularities.

- Three special cases have been included. The first one treats V as a function of $\{\theta, r\}$ only: $V(\theta, r)$. It is convenient, because it allows to build up a physical geodesy without λ-dependence. Even satellite trajectories can be included. It has been dealt with in the dissertations by Gerontopoulos (1978) and Van Gelderen (1991). It has also the advantage to be extendable to the ellipsoidal case without major complications.

  The second special case is two-dimensional cartesian $V(x, z)$. Thus, the field is assumed to be invariant in y-direction: $\frac{\partial^2 V}{\partial y^2} = 0$. This model is very popular in geophysics and applied there at many instances. It permits to demonstrate all principles of the much harder three-dimensional case but leads to very simple Fourier series. It is applied in a very convincing manner throughout in (Turcotte and Schubert, 1982).

  The third special case is derived from the circular cylinder coordinates: $V(\lambda, r)$. It leads to boundary value problems inside and outside a boundary circle in the plane. Again it is extremely simple but allows to demonstrate essential features. It is employed, for example, by Walter (1971).

References