a) The "Plate Carrée map" or "isoparametric mapping" is the simplest cylindrical mapping and has the mapping equations $x = R\Lambda$, $y = R\Phi$. Since $\Lambda \in [-\pi, \pi]$, $\Phi \in [-\pi/2, \pi/2]$ the North-South and East-West-extension, respectively, of the map is $\Delta y = R(\Phi_{North} - \Phi_{South}) = \pi R$ and $\Delta x = R(\Lambda_{East} - \Lambda_{West}) = 2\pi R$, respectively. The width-to-height ratio of the map should thus be $\Delta x / \Delta y = 2$ which is not true, however. Therefore the map cannot be a "Plate Carrée map".

b) Using a "Plate Carrée map" for the determination of areas or area ratios is not suitable because it is not an area preserving mapping. This can be seen from the fact that $\det \underline{J} = R^2 \neq \sqrt{\det \underline{G}} = R^2 \cos\Phi$ or $\Lambda_1 = \dfrac{1}{\cos\Phi} \neq \Lambda_2^{-1} = 1$.

c) The general mapping equations for normal cylindrical mappings are given by the formulas $x = R\Lambda$, $y = Rf(\Phi)$. From this, the extremal distortions Λ_1, Λ_2 can be easily derived: $\Lambda_1 = \sqrt{\dfrac{C_{11}}{G_{11}}} = \dfrac{1}{\cos\Phi}$, $\Lambda_2 = \sqrt{\dfrac{C_{22}}{G_{22}}} = f' = \dfrac{df}{d\Phi}$. By reason of both $\underline{G}$ und $\underline{C}$ being diagonal matrices extremal distortions are along the images of the parameter lines. The equivalence postulate leads to the requirement $\Lambda_1\Lambda_2 = 1 \Leftrightarrow f' = \cos\Phi \Leftrightarrow f = \sin\Phi + c$. In order to achieve y(0)=0 the integration constant c is set to zero, as usual. The final set of mapping equation for our scientific problem of finding the continental-to-ocean area ration is therefore $x = R\Lambda$, $y = R\sin\Phi$. This is nothing else but the cylindrical Lambert equal area mapping of the sphere. Of course, the same result is achieved from the postulate $\det \underline{J} = R^2 f'(\Phi) = \sqrt{\det \underline{G}} = R^2 \cos\Phi \Leftrightarrow f'(\Phi) = \cos\Phi \Leftrightarrow f(\Phi) = \sin\Phi + c$.

a) For the reason that both systems have different origins, different scales and different axes orientation two reasonable proposals are

a_1) the four-parameter (similarity) transformation model (1 scale, 1 rotation, 2 translations)

$$\begin{bmatrix} v \\ u \end{bmatrix} = m \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} + \begin{bmatrix} T_y \\ T_x \end{bmatrix}$$

a_2) the six-parameter (affine) transformation model (4 affine parameters, 2 translations)

$$\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} + \begin{bmatrix} T_y \\ T_x \end{bmatrix}$$

b) $$\underline{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}, \underline{R}_2(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \underline{R}_3(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mathematically positive means that angles are measured/counted counter clockwise as seen from the tip/end of the axis.

c) From "False Easting" E = 729 741.67 m, "False Northing" N = 3 128 844.77 m and Zone = 43, we can first easily determine the reference meridian $L_0 = 6°(\text{Zone} - 30) - 3° = 75°$. Second the distance from the equator is roughly 3 000 km to the North, i.e. 1/3 of a quadrant which makes 30° North. There is only one point in that region: New Delhi.

The exact meaning of the numbers is: The point is located 229 741.67 m East from its reference meridian with longitude $L_0 = 75°$ and 3 128 844.77 m North from the equator.

a) Jacobian matrix $\underline{J}$ and Cauchy-Green-Tensor $\underline{C} = \underline{J}^T \underline{g}\, \underline{J},\ \underline{g} = \underline{I}_2$

$$\underline{J} = R \begin{bmatrix} \frac{\cos\Phi}{\beta + \alpha\cos\Phi} & \frac{-\beta\Lambda\sin\Phi}{(\beta+\alpha\cos\Phi)^2} \\ 0 & \beta+\alpha\cos\Phi \end{bmatrix} = \frac{R}{(\beta+\alpha\cos\Phi)^2} \begin{bmatrix} \cos\Phi(\beta+\alpha\cos\Phi) & -\beta\Lambda\sin\Phi \\ 0 & (\beta+\alpha\cos\Phi)^3 \end{bmatrix},$$

$$\underline{C} = \frac{R^2}{(\beta+\alpha\cos\Phi)^4} \begin{bmatrix} \cos^2\Phi(\beta+\alpha\cos\Phi)^2 & -\beta\Lambda\sin\Phi\cos\Phi(\beta+\alpha\cos\Phi) \\ -\beta\Lambda\sin\Phi\cos\Phi(\beta+\alpha\cos\Phi) & \beta^2\Lambda^2\sin^2\Phi + (\beta+\alpha\cos\Phi)^6 \end{bmatrix}$$

so that extremal distortions appear only along the parameter lines if $\beta = 0$ ("Lambert cylindrical mapping"), since $C_{12}=C_{21}=G_{12}=G_{21}=0$ in that case. Alternatively one can use the argument that if $\underline{g}=\underline{I}_2$ and $x = x(\Lambda), y = y(\Phi)$ or $x = x(\Phi), y = y(\Lambda)$ the term

$$C_{12} = C_{21} = J_{11}J_{12} + J_{21}J_{22} = \frac{\partial x}{\partial\Lambda}\frac{\partial x}{\partial\Phi} + \frac{\partial y}{\partial\Lambda}\frac{\partial y}{\partial\Phi}$$ vanishes.

b) The mapping is of equal-area type if – with given $\underline{G} = R^2 \mathrm{diag}(\cos^2\Phi, 1)$ –

$$\Lambda_1\Lambda_2 = \det(\underline{C}\,\underline{G}^{-1}) = \frac{\det\underline{C}}{\det\underline{G}} = \frac{\det(\underline{J}'\underline{J})}{\det\underline{G}} = \frac{(\det\underline{J})^2}{\det\underline{G}} = 1 \Leftrightarrow \det\underline{J} = \sqrt{\det\underline{G}} = R^2\cos\Phi$$

holds true. Indeed $\det\underline{J} = R^2\cos\Phi$.

c) Conformality is achieved if

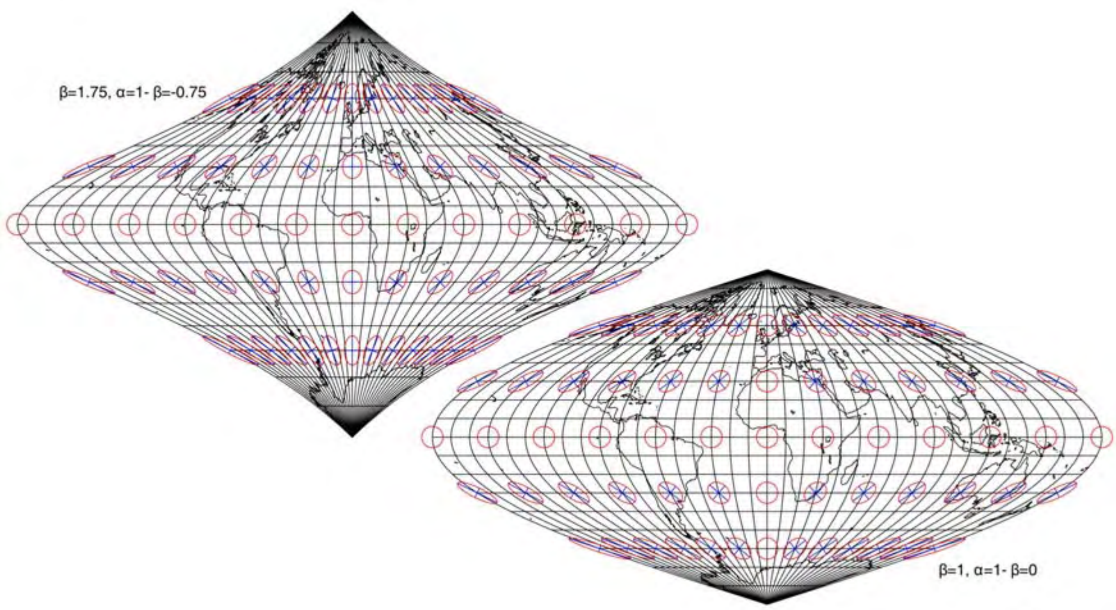
$$\Lambda_1 = \Lambda_2 \Leftrightarrow \left[\mathrm{tr}(\underline{C}\,\underline{G}^{-1})\right]^2 - 4\det(\underline{C}\,\underline{G}^{-1}) = 0$$

$$\overset{\det(\underline{C}\,\underline{G}^{-1})=1}{\Leftrightarrow} \quad \mathrm{tr}(\underline{C}\,\underline{G}^{-1}) = \frac{(\beta+\alpha\cos\Phi)^2 + \beta^2\Lambda^2\sin^2\Phi + (\beta+\alpha\cos\Phi)^6}{(\beta+\alpha\cos\Phi)^4} \overset{!}{=} 2$$

is fulfilled.

(c_1) For the case $\alpha = 1-\beta$ this is true on the equator $\Phi = 0, \forall\,\Lambda$ only and

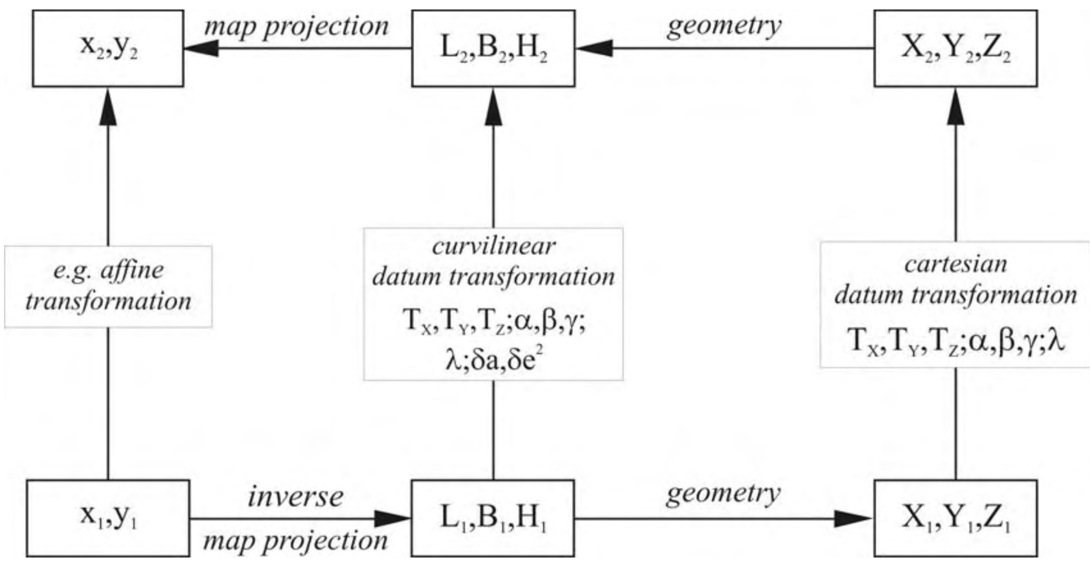
(c_2) for the case $\alpha = 0, \beta = 1$ on the central meridian $\Lambda = 0, \forall\,\Phi$, in addition.



a) Conformal coordinates in the sense of Gauß-Krüger/UTM-coordinates (transverse conformal cylindrical mapping) are based on the (necessary) Cauchy-Riemann differential equations $\partial x/\partial L = \partial y/\partial Q$, $\partial x/\partial Q = -\partial y/\partial L$ and the (sufficient) Laplace differential equations (as derivatives of the Cauchy-Riemann differential equations) between two sets (x,y) and (L,Q) of isometric coordinates; (non isometric) ellipsoidal latitude B is transformed to isometric latitude Q beforehand. As "Ansatz" two homogeneous bivariate polynomials $x(L-L_0=\ell, Q-Q_0=q)$ and $y(L-L_0=\ell, Q-Q_0=q)$ of order n are chosen. The point L_0, Q_0 is a – more or less arbitrarily – chosen origin ("Taylor point") to which conformal coordinates x,y refer. It is a point chosen not too far away from the points to be mapped in order to guarantee convergence of the polynomials. In most practical cases L_0 is an integer value. The coefficients of the homogeneous bivariate polynomials are determined from (i) the requirements of conformality ("Cauchy-Riemann differential equations") and (ii) integrability ("Laplace differential equations"), and (iii) the postulate of an equidistant reference meridian. Before the coefficients are evaluated at the (ellipsoidal latitude B_0 of the) Taylor point a back transformation from isometric to ellipsoidal latitude is necessary.
Gauß-Krüger-coordinate "Northing" is then generated from conformal coordinate x by adding the meridional arc length from the equator to the latitude $B_0(Q_0)$ of the Taylor point. Northing describes the metrical distance of a point from the equator of the underlying ellipsoid-of-revolution. Gauß-Krüger-coordinate "False Easting" is computed from conformal coordinate y by (i) adding a constant (mostly 500 km) to avoid negative numbers for points westerly of L_0 and (ii) computing a reference number from L_0 which is put in front of the number. Thus the arbitrariness of the initial origin is removed.
UTM coordinates are generated from conformal coordinates using identical bivariate polynomials except that both are scaled by an (conventional or optimally chosen) scale factor before. Therefore two meridians west and east of the reference meridian are mapped equidistantly. The reference number computed from L_0 is replaced by a zone number which completes the "False Eastings".

b) Point P is located 34,012 45 km east of the reference meridian with longitude $L_0 = 6° \times (\text{zone number} - 30) - 3° = 15°$ and 4989,99134 km north of the equator.

c)



d) Translation parameters T_X, T_Y, T_Z Rotation parameters: α, β, γ Scale parameter: λ

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \lambda \underline{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} T_X \\ T_Y \\ T_Z \end{pmatrix}, \quad \underline{R} = \underline{R}_3(\gamma)\underline{R}_2(\beta)\underline{R}_1(\alpha)$$

$$\underline{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}, \underline{R}_2(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \underline{R}_3(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) The changed mapping equations are

$$\begin{bmatrix} x \\ y \end{bmatrix} = 2Rf(\Phi_0)\tan\left(\frac{\pi}{4}-\frac{\Phi}{2}\right)\begin{bmatrix} \cos\Lambda \\ \sin\Lambda \end{bmatrix},$$

and the unknown function $f(\Phi_0)$ must be derived from the postulate of an equidistant mapping of the parallel circle $\Phi = \Phi_0$. Multiplying the mapping equations with an arbitrary function f results in a multiplication of the given principal distortions with that function: $\Lambda_1 = \Lambda_2 = \frac{f}{\cos^2\left(\frac{\pi}{4}-\frac{\Phi}{2}\right)}$. An equidistant mapping of the parallel $\Phi = \Phi_0$ is achieved by finding the function from the postulate

$$\Lambda_1(\Phi=\Phi_0) = \Lambda_2(\Phi=\Phi_0) = \frac{f(\Phi_0)}{\cos^2\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right)} = 1$$

$$\Leftrightarrow$$

$$f(\Phi_0) = \cos^2\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right) = \frac{1}{2}(1+\sin\Phi_0) = \frac{\cos\Phi_0}{2\tan\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right)}.$$

However, more easily $f(\Phi_0)$ can be found from the fact that in case of an equidistant mapping the image of the parallel $\Phi = \Phi_0$ must have a radius $r = \sqrt{\left(x|_{\Phi=\Phi_0}\right)^2 + \left(y|_{\Phi=\Phi_0}\right)^2} = 2Rf(\Phi_0)\tan\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right) = R\cos\Phi_0$ in the map. We therefore end up with the modified mapping equations

$$\begin{bmatrix} x \\ y \end{bmatrix} = R(1+\sin\Phi_0)\tan\left(\frac{\pi}{4}-\frac{\Phi}{2}\right)\begin{bmatrix} \cos\Lambda \\ \sin\Lambda \end{bmatrix} = 2R\cos^2\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right)\tan\left(\frac{\pi}{4}-\frac{\Phi}{2}\right)\begin{bmatrix} \cos\Lambda \\ \sin\Lambda \end{bmatrix}.$$

b) The plane (which is parallel to the equator) intersects the sphere at $\Phi = \Phi_0$: It is not a tangential plane.

c) The image of the parallel circle $\Phi=\Phi_0$ has clearly a radius

$$r = \sqrt{\left(x|_{\Phi=\Phi_0}\right)^2 + \left(y|_{\Phi=\Phi_0}\right)^2} = 2R\cos^2\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right)\tan\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right) =$$

$$= 2R\sin\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right) = R\sin\left[2\left(\frac{\pi}{4}-\frac{\Phi_0}{2}\right)\right] =$$

$$= R\sin\left(\frac{\pi}{2}-\Phi_0\right) = R\cos\Phi_0$$

a) First, the model equations – which are non-linear in the unknowns m_x, m_y, μ, ω, c_1 and c_2 – are transformed into linear equations by introducing new variables a, b, d, e with $a := m_x \cos\mu,\ d := m_x \sin\mu,\ b := -m_y \sin\omega,\ e := m_y \cos\omega$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Next, the equations are re-arranged to

$$[x, y] = [1, X, Y] \begin{bmatrix} c_1 & c_2 \\ a & d \\ b & e \end{bmatrix}.$$

We end up with a matrix equation which can easily be solved by inversion of a 3×3 matrix.

$$\underbrace{\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}}_{3\times 2} = \underbrace{\begin{bmatrix} 1 & X_1 & Y_1 \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} c_1 & c_2 \\ a & d \\ b & e \end{bmatrix}}_{3\times 2} \sim \underline{\ell} = \underline{A}\,\underline{\xi},$$

The solution is

$$\begin{bmatrix} \hat{c}_1 & \hat{c}_2 \\ \hat{a} & \hat{d} \\ \hat{b} & \hat{e} \end{bmatrix} = \begin{bmatrix} 1 & X_1 & Y_1 \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{bmatrix}^{-1} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} \sim \hat{\underline{\xi}} = \underline{A}^{-1}\underline{\ell}$$

and the original unknown parameters scales and rotation angles are computed from

$$\hat{\mu} = \arctan\frac{\hat{d}}{\hat{a}},\ \hat{\omega} = \arctan\frac{-\hat{b}}{\hat{e}},\ \hat{m}_x = \sqrt{\hat{a}^2 + \hat{d}^2},\ \hat{m}_y = \sqrt{\hat{b}^2 + \hat{e}^2}\ .$$

b) The inverse coefficient matrix $\underline{A}^{-1}$ is easily computed using elementary operations such as the inversion by subdeterminants:

$$\begin{bmatrix} 1 & -1 & 7 \\ 1 & 1 & 5 \\ 1 & -2 & 6 \end{bmatrix}^{-1} = \frac{1}{6-5-7\times 2-(-5\times 2-6+7)} \begin{bmatrix} 6+5\times 2 & -(6-5) & -2-1 \\ -(-6+14) & 6-7 & -(-2+1) \\ -5-7 & -(5-7) & 2 \end{bmatrix}^{T} =$$

$$= \frac{1}{-13+9} \begin{bmatrix} 16 & -8 & -12 \\ -1 & -1 & 2 \\ -3 & 1 & 2 \end{bmatrix} =$$

$$= -\frac{1}{4} \begin{bmatrix} 16 & -8 & -12 \\ -1 & -1 & 2 \\ -3 & 1 & 2 \end{bmatrix}.$$

Right multiplication with $\underline{\ell}$ results in

$$\begin{bmatrix} \hat{c}_1 & \hat{c}_2 \\ \hat{a} & \hat{d} \\ \hat{b} & \hat{e} \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 16 & -8 & -12 \\ -1 & -1 & 2 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 7 & 5 \\ 4 & 4 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -24 & 24 \\ -4 & -4 \\ 0 & -8 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 1 & 1 \\ 0 & 2 \end{bmatrix},$$

and scale factors and rotations angles can be found to be

$$\hat{\mu} = \arctan\frac{\hat{d}}{\hat{a}} = \arctan(1) = 45°,\ \hat{\omega} = \arctan\frac{-\hat{b}}{\hat{e}} = \arctan(0) = 0°$$

$$\hat{m}_x = \sqrt{\hat{a}^2 + \hat{d}^2} = \sqrt{2},\ \hat{m}_y = \sqrt{\hat{b}^2 + \hat{e}^2} = 2.$$

c) Missing coordinates X_4,Y_4 of point 4 are computed from inversion of the model equations:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1}\begin{bmatrix} x - c_1 \\ y - c_2 \end{bmatrix}$$

$$\begin{bmatrix} \hat{X}_4 \\ \hat{Y}_4 \end{bmatrix} = \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{d} & \hat{e} \end{bmatrix}^{-1}\begin{bmatrix} x_4 - \hat{c}_1 \\ y_4 - \hat{c}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$