Lecture Notes

Geodesy and Geoinformatics

part: Geodesy

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These are lecture notes in progress. Please contact me (sneeuw@gis.uni-stuttgart.de) for remarks, errors, suggestions, etc.

Contents

1. Introduction

1.1. Physical Geodesy

Geodesy aims at the determination of the geometrical and physical shape of the Earth and its orientation in space. The branch of geodesy that is concerned with determining the physical shape of the Earth is called *physical geodesy*. It does interact strongly with the other branches, though, as will be seen later.

Physical geodesy is different from other geomatics disciplines in that it is concerned with field quantities: the scalar potential field or the vectorial gravity and gravitational fields. These are continuous quantities, as opposed to point fields, networks, pixels, etc., which are discrete by nature.

Gravity field theory uses a number of tools from mathematics and physics:

- Newtonian gravitation theory (relativity is not required for now)
- Potential theory
- Vector calculus
- Special functions (Legendre)
- Partial differential equations
- Boundary value problems
- Signal processing

Gravity field theory is interacting with many other disciplines. A few examples may clarify the importance of physical geodesy to those disciplines. The Earth science disciplines are rather operating on a global scale, whereas the engineering applications are more local. This distinction is not fundamental, though.

1.2. Links to Earth sciences

Oceanography. The Earth's gravity field determines the geoid, which is the equipotential surface at mean sea level. If the oceans would be at rest—no waves, no currents, no tides—the ocean surface would coincide with the geoid. In reality it deviates by up to 1 m. The difference is called *sea surface topography*. It reflects the dynamical equilibrium in the oceans. Only large scale currents can sustain these deviations.

The sea surface itself can be accurately measured by radar altimeter satellites. If the geoid would be known up to the same accuracy, the sea surface topography and consequently the global ocean circulation could be determined. The problem is the insufficient knowledge of the marine geoid.

Geophysics. The Earth's gravity field reflects the internal mass distribution, the determination of which is one of the tasks of geophysics. By itself gravity field knowledge is insufficient to recover this distribution. A given gravity field can be produced by an infinity of mass distributions. Nevertheless, gravity is is an important constraint, which is used together with seismic and other data.

As an example, consider the gravity field over a volcanic island like Hawaii. A volcano by itself represents a geophysical anomaly already, which will have a gravitational signature. Over geologic time scales, a huge volcanic mass is piled up on the ocean sphere. This will cause a bending of the ocean floor. Geometrically speaking one would have a cone in a bowl. This bowl is likely to be filled with sediment. Moreover the mass load will be supported by buoyant forces within the mantle. This process is called *isostasy*. The gravity signal of this whole mass configuration carries clues to the density structure below the surface.

Normal mode seismology. TBD

Geology. Different geological formations have different density structures and hence different gravity signals. One interesting example of this is the Chicxulub crater, partially on the Yucatan peninsula (Mexico) and partially in the Gulf of Mexico. This crater with a diameter of 180 km was caused by a meteorite impact, which occurred at the K-T boundary (cretaceous-tertiary) some 66 million years ago. This impact is thought to have caused the extinction of dinosaurs. The Chicxulub crater was discovered by careful analysis of gravity data.

Hydrology. Minute changes in the gravity field over time—after correcting for other time-variable effects like tides or atmospheric loading—can be attributed to changes in hydrological parameters: soil moisture, water table, snow load. For static gravimetry these are usually nuisance effects. Nowadays, with precise satellite techniques, hydrology is one of the main aims of spaceborne gravimetry. Despite a low spatial resolution, the results of satellite gravity missions may be used to constrain basin-scale hydrological parameters.

Glaciology and sea level. The behaviour of the Earth's ice masses is a critical indicator of global climate change and global sea level behaviour. Thus, monitoring of the melting of the Greenland and Antarctica ice caps is an important issue. The ice caps are huge mass loads, sitting on the Earth's crust, which will necessarily be depressed. Melting causes a rebound of the crust. This process is still going on since the last Ice Age, but there is also an instant effect from melting taking place right now. The change in surface ice contains a direct gravitational component and an effect, due to the uplift. Therefore, precise gravity measurements carry information on ice melting and consequently on sea level rise.

1.3. Applications in engineering

Geophysical prospecting. Since gravity contains information on the subsurface density structure, gravimetry is a standard tool in the oil and gas industry (and other mineral resources for that matter). It will always be used together with seismic profiling, test drilling and magnetometry. The advantages of gravimetry over these other techniques are:

- relatively inexpensive,
- non destructive (one can easily measure inside buildings),
- compact equipment, e.g. for borehole measurements

Gravimetry is used to localize salt domes or fractures in layers, to estimate depth, and in general to get a first idea of the subsurface structure.

Geotechnical Engineering. In order to gain knowledge about the subsurface structure, gravimetry is a valuable tool for certain geotechnical (civil) engineering projects. One can think of determining the depth-to-bedrock for the layout of a tunnel. Or making sure no subsurface voids exist below the planned building site of a nuclear power plant.

For examples, see the (micro-)gravity case histories and applications on:
`http://www.geop.ubc.ca/ubcgif/casehist/index.html`, or
`http://www.esci.keele.ac.uk/geophysics/Research/Gravity/`.

Geomatics Engineering. Most surveying observables are related to the gravity field.

i) After leveling a **theodolite** or a **total station**, its vertical axis is automatically aligned with the local gravity vector. Thus all measurements with these instruments are referenced to the gravity field—they are in a local astronomic frame. To convert them to a geodetic frame the deflection of the vertical (ξ,η) and the perturbation in azimuth (ΔA) must be known.

ii) The line of sight of a **level** is tangent to the local equipotential surface. So levelled height differences are really physical height differences. The basic quantity of physical heights are the potentials or the potential differences. To obtain precise height differences one should also use a gravimeter:

$$\Delta W = \int_A^B \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{x} = \int_A^B g \mathrm{d}h \approx \sum_i g_i \Delta h_i \,.$$

The Δh_i are the levelled height increments. Using gravity measurements g_i along the way gives a geopotential difference, which can be transformed into a physical height difference, for instance an orthometric height difference.

iii) **GPS positioning** is a geometric technique. The geometric GPS heights are related to physically meaningful heights through the geoid or the quasi-geoid:

$$\begin{aligned} h &= H + N &&= \text{orthometric height} + \text{geoid height}, \\ h &= H^{\mathrm{n}} + \zeta &&= \text{normal height} + \text{quasi-geoid height}. \end{aligned}$$

In geomatics engineering, GPS measurements are usually made over a certain baseline and processed in differential mode. In that case, the above two formulas become $\Delta h = \Delta N + \Delta H$, etc. The geoid difference between the baseline's endpoints must therefore be known.

iv) The basic equation of **inertial surveying** is $\ddot{\boldsymbol{x}} = \boldsymbol{a}$, which is integrated twice to provide the trajectory $\boldsymbol{x}(t)$. The equation says that the kinematic acceleration equals the specific force vector $\boldsymbol{a}$: the sum of all forces (per unit mass) acting on a proof mass). An inertial measurement unit, though, measures the sum of kinematic acceleration and gravitation. Thus the gravitational field must be corrected for, before performing the integration.

2. Approximation 1: the sphere

The Earth's surface is a complicated manifold. For many purposes in surveying, navigation and several geosciences, a spherical description is more than sufficient. With a flattening in the order of 10^{-3} a spherical approximation implies errors less than 1 %. For geodetic applications in which this error level is unacceptable, an ellipsoid of revolution is used as a higher quality approximation. This chapter provides tools to perform calculations on these surfaces.

Remark 2.1 *In this chapter, the symbol ϕ will be used for the* geocentric *latitude.*

2.1. Basic spherical geometry

The sphere can be described in a number of ways, see fig. 2.1:

geometrically as the set of points with constant distance (or radius) to a focal point at the centre, leading to the following algebraic formulation.

algebraically (implicit) $x^2 + y^2 + z^2 = R^2 \Longrightarrow \frac{x^2}{R^2} + \frac{y^2}{R^2} + \frac{z^2}{R^2} = 1$

parametrically (explicit)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\phi\cos\lambda \\ r\cos\phi\sin\lambda \\ r\sin\phi \end{pmatrix} \Longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \lambda = \arctan\frac{y}{x} \\ \phi = \arcsin\frac{z}{r} \end{cases}$$

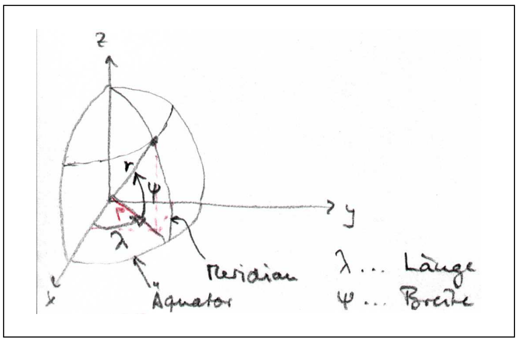

Figure 2.1: Spherical geometry. (please read ϕ for ψ)

2.2. From planar to spherical trigonometry

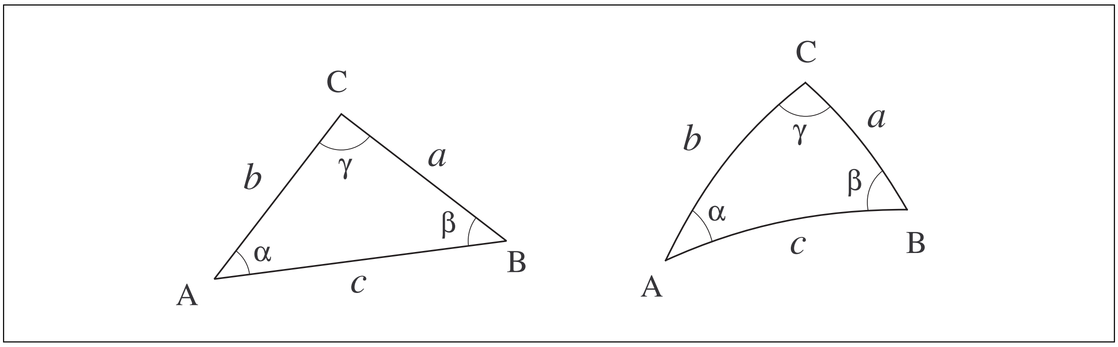

Figure 2.2.: Planar and spherical triangle.

When going from the plane to the sphere many trigonometric relationships between angles and sides are similar. One must be careful, though. In plane trigonometry, triangle sides are line segments, measured in linear units. On the sphere, however, sides are *great circle* segments, or rather angles, expressed in angular units. They may be converted to linear units, e.g., by $s_a = aR$, with R the spherical radius. The following relationships exist—mostly in parallel—between planar and spherical trigonometry: Großkreis

	planar	**spherical**
angles:	$\alpha + \beta + \gamma = 180°$	$\alpha + \beta + \gamma = 180° + \varepsilon$
area:	$2s = a + b + c$	$2s = a + b + c$
	$A = \sqrt{s(s-a)(s-b)(s-c)}$	$\tan \frac{1}{4}\varepsilon = \sqrt{\tan \frac{s}{2} \tan \frac{s-a}{2} \tan \frac{s-b}{2} \tan \frac{s-c}{2}}$
	(Heron's formula)	(l'Huilier's formula)
		$A = R^2 \varepsilon$
sine:	$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$	$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c}$
cosine:	$a^2 = b^2 + c^2 - 2bc\cos\alpha$	$\cos a = \cos b \cos c + \sin b \sin c \cos\alpha$

Further cosine formulas and sine-cosine formulas are obtained by cyclic permutation $a \to b \to c \to a \to ...$ and $\alpha \to \beta \to \gamma \to \alpha \to ...$

The quantity ε is called the *spherical excess*. According to the above formula, the sum of angles in a spherical triangle is more than 180°. How much more, depends on the area of the triangle. The formula $A = R^2\varepsilon$ actually tells us that ε is the solid (geo-)centric angle, subtended by the spherical triangle. The unit of a solid angle is steradian.

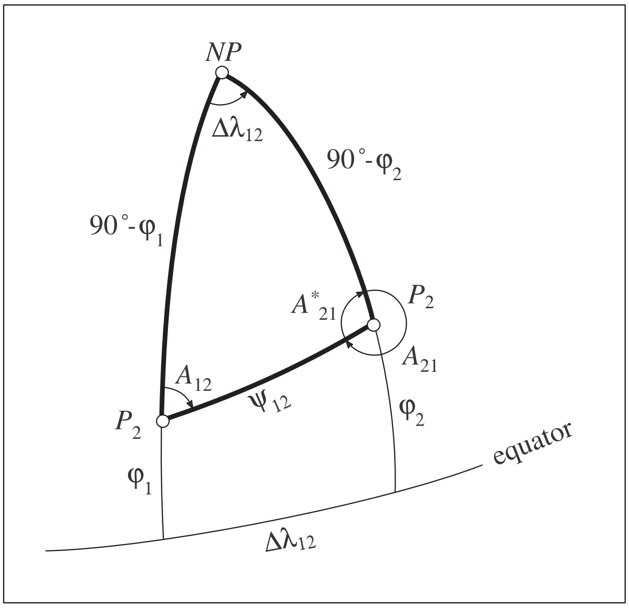

Figure 2.3: The polar spherical triangle.

Remark 2.2 *Consider the extreme spherical triangle of the following 3 points: Northpole, intersection of Greenwich meridian and equator, and the point on the equator at* 90° *longitude. All of the angles in this triangle are right angles. Thus* $\alpha + \beta + \gamma = 270°$*, i.e.* $\varepsilon = 90°$.

Exercise 2.1 *Determine the sides of the triangle in remark 2.2 and check the validity of* all *above spherical trigonometric formulas.*

The discussion of the direct and inverse problems in the following sections is based on the so-called *polar spherical triangle*, see fig. 2.3.

2.3. The direct problem

Anfangswertproblem

The direct problem is defined as the following *initial value problem*:

Given: ϕ_1 and λ_1 of the first point
ψ_{12} and A_{12} between the first and second point
Find: ϕ_2 and λ_2 of the second point
and the inverse azimuth A_{21}

Determination of ϕ_2

From the spherical cosine formula:

$$\cos(90° - \phi_2) = \cos(90° - \phi_1)\cos\psi_{12} + \sin(90° - \phi_1)\sin\psi_{12}\cos A_{12}$$

$$\Rightarrow \boxed{\sin\phi_2 = \sin\phi_1 \cos\psi_{12} + \cos\phi_1 \sin\psi_{12} \cos A_{12}}$$

Determination of λ_2

From the spherical cosine formula:

$$\cos\psi_{12} = \cos(90^\circ - \phi_1)\cos(90^\circ - \phi_2) + \sin(90^\circ - \phi_1)\sin(90^\circ - \phi_2)\cos\Delta\lambda_{12}$$

$$\Rightarrow \cos\Delta\lambda_{12} = \frac{\cos\psi_{12} - \sin\phi_1 \sin\phi_2}{\cos\phi_1 \cos\phi_2}$$

$$\Rightarrow \boxed{\lambda_2 = \lambda_1 + \Delta\lambda_{12}}$$

Determination of reverse azimuth A_{21}

From the spherical cosine formula:

$$\begin{aligned}\cos(90^\circ - \phi_1) &= \cos(90^\circ - \phi_2)\cos\psi_{12} + \sin(90^\circ - \phi_2)\sin\psi_{12}\cos A^*_{21}\\ \Rightarrow \cos A^*_{21} &= \frac{\sin\phi_1 - \sin\phi_2 \cos\psi_{12}}{\cos\phi_2 \sin\psi_{12}}\end{aligned}$$

From the spherical sine formula:

$$\frac{\sin A^*_{21}}{\sin(90^\circ - \phi_1)} = \frac{\sin\Delta\lambda_{12}}{\sin\psi_{12}} \quad \Rightarrow \quad \sin A^*_{21} = \frac{\sin\Delta\lambda_{12}\cos\phi_1}{\sin\psi_{12}}$$

$$\Rightarrow \boxed{A_{21} = 360^\circ - \arctan\frac{\sin A^*_{21}}{\cos A^*_{21}}}$$

2.4. The inverse problem

The inverse problem is defined as the following boundary value problem: Randwertproblem

Given: ϕ_1 and λ_1 of the first point
ϕ_2 and λ_2 of the second point
Find: ψ_{12} between the first and second point
The azimuths A_{12} and A_{21} in both end points

Determination of spherical distance ψ_{12}

From the spherical cosine formula:

$$\cos\psi_{12} = \cos(90^\circ - \phi_1)\cos(90^\circ - \phi_2) + \sin(90^\circ - \phi_1)\sin(90^\circ - \phi_2)\cos\Delta\lambda_{12}$$

$$\Rightarrow \boxed{\cos\psi_{12} = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda_{12}}$$

Determination of azimuth A_{12}

From the spherical cosine formula:

$$\begin{aligned}\cos(90^\circ - \phi_2) &= \cos(90^\circ - \phi_1)\cos\psi_{12} + \sin(90^\circ - \phi_1)\sin\psi_{12}\cos A_{12} \\ \Rightarrow \cos A_{12} &= \frac{\sin\phi_2 - \sin\phi_1\cos\psi_{12}}{\cos\phi_1 \sin\psi_{12}}\end{aligned}$$

From the spherical sine formula:

$$\frac{\sin A_{12}}{\sin(90^\circ - \phi_2)} = \frac{\sin\Delta\lambda_{12}}{\sin\psi_{12}} \quad \Rightarrow \quad \sin A_{12} = \frac{\sin\Delta\lambda_{12}\cos\phi_2}{\sin\psi_{12}}$$

$$\Rightarrow \boxed{A_{12} = \arctan\frac{\sin A_{12}}{\cos A_{12}}}$$

Determination of azimuth A_{21}

From the spherical cosine formula:

$$\cos(90^\circ - \phi_1) = \cos(90^\circ - \phi_2)\cos\psi_{12} + \sin(90^\circ - \phi_2)\sin\psi_{12}\cos A^*_{21} \Rightarrow$$

$$\cos A^*_{21} = \frac{\sin\phi_1 - \sin\phi_2\cos\psi_{12}}{\cos\phi_2\sin\psi_{12}}$$

From the spherical sine formula:

$$\frac{\sin A^*_{21}}{\sin(90^\circ - \phi_1)} = \frac{\sin\Delta\lambda_{12}}{\sin\psi_{12}} \quad \Rightarrow \quad \sin A^*_{21} = \frac{\sin\Delta\lambda_{12}\cos\phi_1}{\sin\psi_{12}}$$

$$\Rightarrow \boxed{A_{12} = 360^\circ - \arctan\frac{\sin A^*_{21}}{\cos A^*_{21}}}$$

Remark 2.3 *In the above derivations extra effort has been put into defining the angles in the right quadrant by determining an angle both with a sine-rule and a cosine-rule. In many cases, in which the quadrant is clear, simpler formulas like the sine formulas would be sufficient.*

3. Approximation 2: the ellipsoid

Remark 3.1 *In this chapter, the symbol ϕ will be used for the* geodetic *or ellipsoidal latitude.*

3.1. Basic ellipsoidal geometry

The ellipsoid is described in several ways:

geometrically The ellipse is defined as the set of points whose sum of distances to two *foci* is constant. This definition provides a curve in two-dimensional space. The *bi-axial* ellipsoid in 3D space is the result of rotating the ellipse around one of its axes. Brennpunkte zweiachsig

Inspection of fig. 3.1, in which we choose a point on the major axis (left panel), tells us that this sum must be $(a+x)+(a-x)=2a$, the length of the major axis. The quantity a is called the *semi-major axis*. lange Halbachse

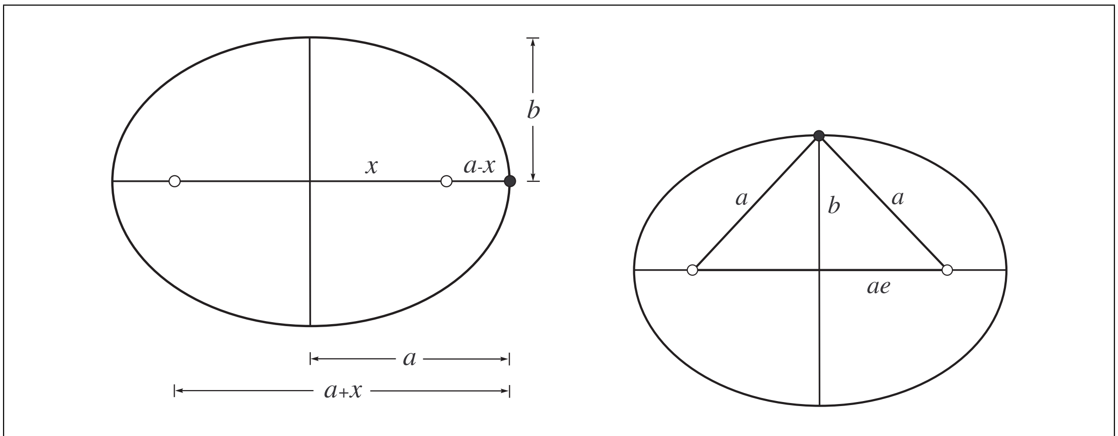

Figure 3.1.: Planar geometry of the ellipse.

But then, for a point on the minor axis, see right panel, we have a symmetrical configuration. The distance from this point to each of the foci is a. The length b is called the *semi-minor axis*. Knowing both axes, we can express the distance to focus and centre of the ellipse. It is $\sqrt{a^2-b^2}$. Usually it is expressed as a kurze Halbachse proportion e of the semi-major axis a:

$$(ae)^2 + b^2 = a^2 \Longrightarrow e^2 = \frac{a^2 - b^2}{a^2}\ , \text{ or } b = \sqrt{1 - e^2}\ a\,.$$

Exzentrizität

The proportionality factor e is called the *eccentricity*; the out-of-centre distance ae is known as the *linear eccentricity*.

algebraically (implicit) In case the axis of symmetry is the z-axis:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1\,.$$

One can obtain a 2D again by the following substitution:

$$\begin{cases} x = p\cos\lambda \\ y = p\sin\lambda \end{cases} \Longrightarrow \frac{p^2}{a^2} + \frac{z^2}{b^2} = 1\,,$$

in which $p = \sqrt{x^2 + y^2}$ can be considered the horizontal coordinate in the meridian plane.

parametrically (explicit) For points *on* the ellipsoid the transformation from ellipsoidal to Cartesian coordinates reads:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} N(\phi)\cos\phi\cos\lambda \\ N(\phi)\cos\phi\sin\lambda \\ N(\phi)(1 - e^2)\sin\phi \end{pmatrix}\ , \text{ with:} \quad N(\phi) = \frac{a}{\sqrt{1 - e^2\sin^2\phi}} \tag{3.1a}$$

For points above the ellipsoidal surface, we have to add the ellipsoidal height h in normal direction as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (N + h)\cos\phi\cos\lambda \\ (N + h)\cos\phi\sin\lambda \\ (N(1 - e^2) + h)\sin\phi \end{pmatrix} \tag{3.1b}$$

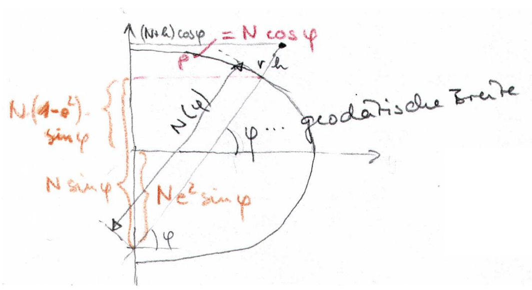

Figure 3.2.: Ellipsoidal geometry.

A closed analytical solution for the reverse transformation from Cartesian to geodetic coordinates does exist. Here, however, we will simply apply an iteration. First off, longitude can be determined by: $\tan\lambda = \frac{y}{x}$. But geodetic latitude and height must be solved iteratively together. To that end we introduce the coordinate p again (distance to z-axis):

$$p = \sqrt{x^2 + y^2} = (N + h)\cos\phi$$

iteration equation 1: $$h = \frac{p}{\cos\phi} - N(\phi)$$

$$z = (N(1 - e^2) + h)\sin\phi \quad \Longrightarrow \quad \frac{z}{p} = \frac{N(1 - e^2) + h}{N + h}\tan\phi$$

iteration equation 2: $$\phi = \arctan\left(\frac{z}{p}\frac{N + h}{N(1 - e^2) + h}\right)$$

With the two above equations, the iteration runs as follows:

- Starting value $i = 0$: $h_0 = 0$ (just assume point on surface, if no better information available).
- Starting latitude: $\phi_0 = \arctan(\frac{z}{p}\frac{1}{(1-e^2)})$ from iteration equation 2.
- $N(\phi_0) = \ldots$
- $h_{i+1} = \frac{p}{\cos\phi_i} - N(\phi_i)$ from iteration equation 1.
- $\phi_{i+1} = \arctan\left(\frac{z}{p}\frac{N(\phi_i)+h_i}{N(\phi_i)(1-e^2)+h-i}\right)$ from iteration equation 2 again.
- $N(\phi_{i+1}) =$ and so on.
- Iteration until convergence is achieved
  $|h_{i+1} - h_i| < \varepsilon_h$
  $|\phi_{i+1} - \phi_i| < \varepsilon_\phi$

Geodetic and geocentric latitudes From the implicit formulation of the ellipsoid, we can derive the surface normal vector simply by taking the gradient:

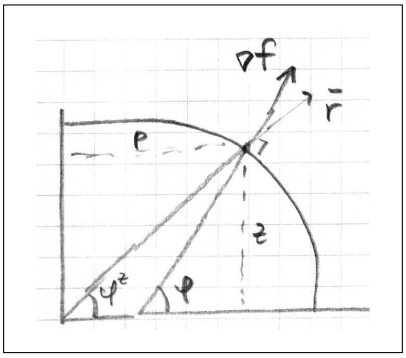

Figure 3.3: Normal vector ∇f vs. radial direction $\boldsymbol{r}$ and link between geodetic latitude ϕ and geocentric latitude ϕ^z

3D	**2D**
$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 = f(x,y,z)$	$\frac{p^2}{a^2} + \frac{z^2}{b^2} = 1 = f(p,z)$
$\nabla f = 2 \begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{a^2} \\ \frac{z}{b^2} \end{pmatrix}$	$\nabla f = 2 \begin{pmatrix} \frac{p}{a^2} \\ \frac{z}{b^2} \end{pmatrix}$
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} N\cos\phi\cos\lambda \\ N\cos\phi\sin\lambda \\ N(1-e^2)\sin\phi \end{pmatrix}$	$\begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} N\cos\phi \\ N(1-e^2)\sin\phi \end{pmatrix}$

From fig. 3.3 the link between geocentric and geodetic latitude becomes clear:

$$\left.\begin{array}{l} \tan\phi^z = \frac{z}{p} \text{ (see figure)} \\ \tan\phi = \frac{z}{b^2} : \frac{p}{a^2} = \frac{a^2 z}{b^2 p} \text{ (from } \nabla f) \end{array}\right\} \Longrightarrow \tan\phi^z = \frac{b^2}{a^2}\tan\phi = (1-e^2)\tan\phi \equiv \frac{z}{p}\,.$$

3.2. Curvature

Sphere An infinitesimal arc length ds on the sphere is related to its infinitesimal central angle simply by multiplying by the sphere's radius R, see fig. 3.4:

$$\mathrm{d}s = R\mathrm{d}\psi\,.$$

This is more or less the translation of dψ in angular measure into linear measure. However, it leads to a more fundamental concept, as the quantity

$$\rho = \frac{1}{R} = \frac{\mathrm{d}\psi}{\mathrm{d}s}$$

Krümmung
Krümmungsradius

is called the *curvature*. The radius R is known as the *radius of curvature*. In general, the

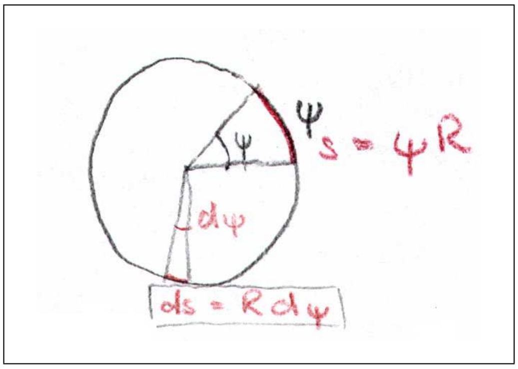

Figure 3.4: Finite and infinitesimal arc length on the sphere.

curvature of a surface is a local quantity, that is, it depends on position. On the sphere, though, curvature is constant. Thus, in 2 we might have added *surface of constant curvature* as a definition of the sphere.

Ellipsoid On the ellipsoid, on the other hand, the curvature is a local measure. To be more precise:

$$\rho = \rho(\phi, \alpha)\,,$$

that is, the curvature is latitude and direction dependent. It is a function of latitude ϕ and on *azimuth* α. At every point on the ellipsoid there will be a direction in which the curvature is maximal and a direction in which it is minimal. Each direction spans up a surface through the local normal vector. Such surfaces are called *normal sections*, see fig. 3.6.

Azimut

Normalschnitte

As might be expected, the two extremes in curvature take place

i) in the meridian section, and

Meridianschnitt

ii) in the prime vertical normal section, which is perpendicular to the meridian section and tangent to the local latitude circle.

Note that the plane through a latitude circle by itself is not a normal section.

Let us consider the curvature and its variations in the meridian and in the equator. The latitude dependence is obvious from fig. 3.5 (left panel). At the equator, the smaller circle fits the ellipse in an optimal way. Its radius is the radius of curvature. It is clear that this radius of curvature is smaller than the semi-major axis a. At the pole, though, the best fitting circle has the largest possible radius, larger than a. Thus the curvature at the pole, $\rho(\phi = 90°)$, is minimum.

At the pole, no directional dependence can exist, as all meridian planes are normal sections. At the equator, though, there will be a clear difference in curvature between meridian plane (as discussed above) and in the equator plane. The equatorial normal section of the ellipsoid is a circle, see fig. 3.5. The radius of curvature at the equator in East-West direction is therefore a and the curvature $\rho(\phi = 0°, \alpha = 90°) = 1/a$. In the previous paragraph, we already concluded that the radius of curvature at the equator in

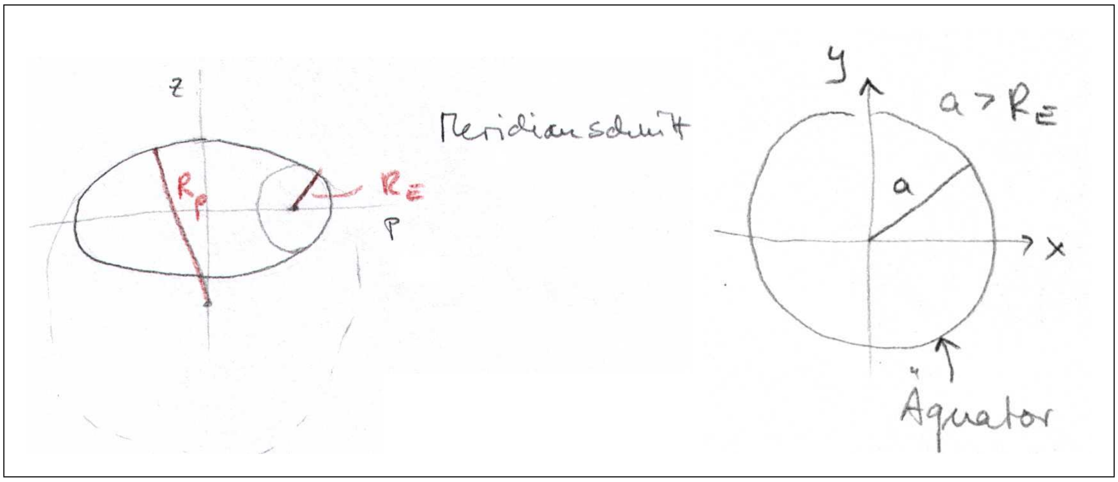

Figure 3.5.: Latitude dependence of curvature in the meridian plane (left) and azimuth dependence at the equator (right).

North-South direction was smaller than a.

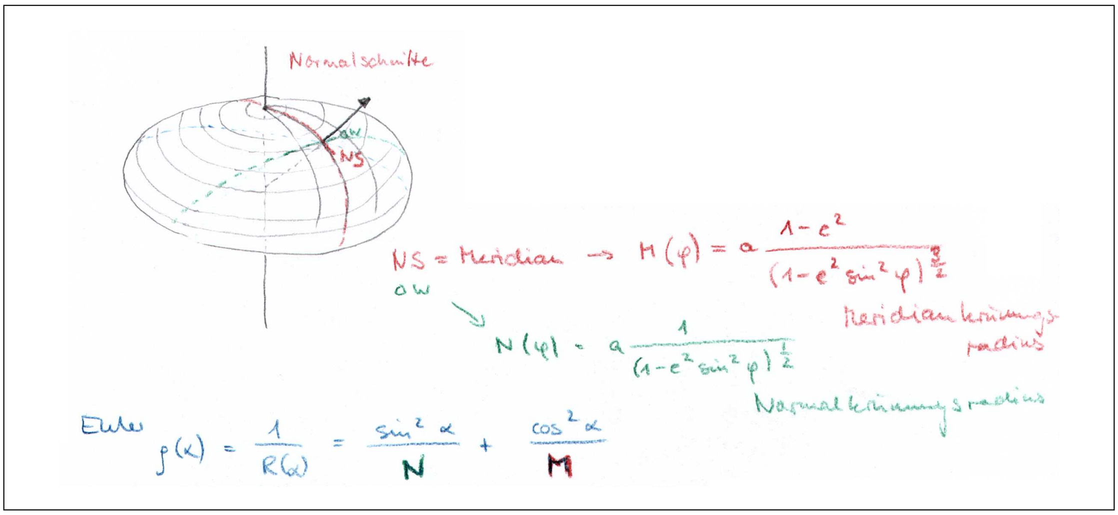

Figure 3.6.

Main radii of curvature This behaviour is not only valid at the equator. At every latitude we will see the minimum radius of curvature (and hence the maximum curvature) in the meridian plane and the maximum radius of curvature in the prime vertical normal section. They are known, respectively, as the *meridian radius of curvature* $M(\phi)$ and normal radius of curvature $N(\phi)$. The latter radius is exactly the quantity that we know already from (3.1). The corresponding equations and some examples are given in the following table.

Meridiankrümmungsradius
Normalkrümmungsradius

	in meridian	in prime vertical
general	$M(\phi) = a\dfrac{1-e^2}{(1-e^2\sin^2\phi)^{3/2}}$	$N(\phi) = a\dfrac{1}{(1-e^2\sin^2\phi)^{1/2}}$
at equator	$M(0°) = a(1-e^2)$	$N(0°) = a$
at pole	$M(90°) = \dfrac{a}{\sqrt{1-e^2}}$	$N(90°) = \dfrac{a}{\sqrt{1-e^2}}$

The table indeed confirms that the smallest radius of curvature is in North-South direction: $M(0°) < N(0°)$. Moreover, at the poles there is no azimuth dependence: $M(90°) = N(90°)$.

Gauss curvature The radius of a best fitting sphere at a certain latitude is the Gauss radius of curvature:

$$R_{\mathrm{G}} = \sqrt{MN} = \frac{a\sqrt{1-e^2}}{1-e^2\sin^2\phi} .$$

Mean curvature The mean curvature is defined by:

$$\rho_{\mathrm{M}} = \frac{1}{R_{\mathrm{M}}} = \frac{1}{2}\left(\frac{1}{M} + \frac{1}{N}\right) .$$

Curvature in arbitrary direction The mathematician Euler developed a formula that relates the curvatures in North-South direction $\rho(\alpha = 0°)$ and in East-West direction $\rho(\alpha = 90°)$ to the curvature in arbitrary direction:

$$\rho(\alpha) = \frac{1}{R_\alpha} = \frac{\sin^2\alpha}{N} + \frac{\cos^2\alpha}{M} . \tag{3.2}$$

3.3. The direct and inverse geodetic problem on the ellipsoid

The shortest path between two points on a curved surface is called a *geodesic*. Solving the direct and inverse geodetic problem on the ellipsoid would require finding and describing geodesics on the ellipsoid. This is a mathematically demanding topic, particularly if analytical solutions are attempted. To exemplify the level of complexity on the ellipsoid, it is remarked that a geodesic is in general not a closed curve, like the great circle on the sphere. It suffices to say that the geodesic is described by a set of three coupled ordinary differential equations, that may be solved numerically.

geodätische Linie

Meridian arc A *meridian arc* s is a special geodesic. It is described by a single differential equation: Meridianbogen

$$\frac{\mathrm{d}s}{\mathrm{d}\phi} = M(\phi)\,,$$

which is of course the reverse of the definition of a differential arc length (compare the spherical case):

$$\mathrm{d}s = M(\phi)\mathrm{d}\phi\,.$$

Therefore, the meridian arc length between two points at different latitudes is

$$s_{1,2} = \int_1^2 \mathrm{d}s = \int_{\phi_1}^{\phi_2} M(\phi)\mathrm{d}\phi\,,$$

which can be evaluated by numerical quadrature.

4. Approximation 3: the geoid

4.1. Newtonian gravitation

In 1687 Newton[1] published his *Philosophiae naturalis principia mathematica*, or *Principia* in short. The Latin title can be translated as *Mathematical principles of natural philosophy*, in which *natural philosophy* can be read as *physics*. Although Newton was definitely not the only physicist working on gravitation in that era, his name is nevertheless remembered and attached to gravity because of the *Principia*. The greatness of this work lies in the fact that Newton was able to bring empirical observations on a mathematical footing and to explain in a unifying manner many natural phenomena:

- planetary motion (in particular elliptical motion, as discovered by Kepler[2]),
- free fall, e.g. the famous apple from the tree,
- tides,
- equilibrium shape of the Earth.

Newton made fundamental observations on gravitation:

- The force between two attracting bodies is proportional to the individual masses.
- The force is inversely proportional to the square of the distance.
- The force is directed along the line connecting the two bodies.

Mathematically, the first two are translated into:

$$F_{12} = G\frac{m_1 m_2}{r_{12}^2}\,, \tag{4.1}$$

in which G is a proportionality factor. It is called the *gravitational constant* or *Newton constant*. It has a value of $G = 6.672 \cdot 10^{-11}\,\mathrm{m^3s^{-2}kg^{-1}}$ (or $\mathrm{N\,m^2\,kg^{-2}}$).

Remark 4.1 (mathematical model of gravitation) *Soon after the publication of the* Principia *Newton was strongly criticized for his law of gravitation, e.g. by his contemporary*

[1] Sir Isaac Newton (1642–1727).

[2] Johannes Kepler (1571–1630), German astronomer and mathematician; formulated the famous laws of planetary motion: *i)* orbits are ellipses with Sun in one of the foci, *ii)* the areas swept out by the line between Sun and planet are equal over equal time intervals (area law), and *iii)* the ratio of the cube of the semi-major axis and the square of the orbital period is constant (or $n^2a^3 = GM$).

Huygens. Equation (4.1) implies that gravitation acts at a distance, and that it acts instantaneously. Such action is unphysical in a modern sense. For instance, in Einstein's relativity theory no interaction can be faster than the speed of light. However, Newton did not consider his formula (4.1) as some fundamental law. Instead, he saw it as a convenient mathematical description. As such, Newton's law of gravitation is still a viable model for gravitation in physical geodesy.

Equation (4.1) is symmetric: the mass m_1 exerts a force on m_2 and m_2 exerts a force of the same magnitude but in opposite direction on m_1. From now on we will be interested in the gravitational field generated by a single test mass. For that purpose we set $m_1 := m$ and we drop the indices. The mass m_2 can be an arbitrary mass at an arbitrary location. Thus we eliminate m_2 by $a = F/m_2$. The gravitational attraction a of m becomes:

$$a = G\frac{m}{r^2}\,, \tag{4.2}$$

in which r is the distance between mass point and evaluation point. The gravitational attraction has units m/s^2. In geodesy one often uses the unit Gal, named after Galileo[3]:

$$\begin{aligned} 1\,\mathrm{Gal} &= 10^{-2}\,\mathrm{m/s^2} = 1\,\mathrm{cm/s^2} \\ 1\,\mathrm{mGal} &= 10^{-5}\,\mathrm{m/s^2} \\ 1\,\mu\mathrm{Gal} &= 10^{-8}\,\mathrm{m/s^2}\,. \end{aligned}$$

Remark 4.2 (kinematics vs. dynamics) *The gravitational attraction is not an acceleration. It is a dynamical quantity: force per unit mass or specific force. Accelerations on the other hand are kinematic quantities.*

Vectorial attraction of a point mass

The gravitational attraction works along the line connecting the point masses. In this symmetrical situation the attraction at point 1 is equal in size, but opposite in direction, to the attraction at point 2: $\boldsymbol{a}_{12} = -\boldsymbol{a}_{21}$. This corresponds to Newton's law: *action* $= -$*reaction*.

In case we have only one point mass m, located in $\boldsymbol{r}_1$, whose attraction is evaluated in point $\boldsymbol{r}_2$, this symmetry is broken. The vector $\boldsymbol{a}$ is considered to be the corresponding attraction.

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}_2 - \boldsymbol{r}_1 = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}\,, \text{ and } r = |\boldsymbol{r}| \\ \boldsymbol{a} &= -G\frac{m}{r^2}\boldsymbol{e}_{12} = -G\frac{m}{r^2}\frac{\boldsymbol{r}}{r} = -G\frac{m}{r^3}\boldsymbol{r} \end{aligned}$$

[3]Galileo Galilei (1564–1642).

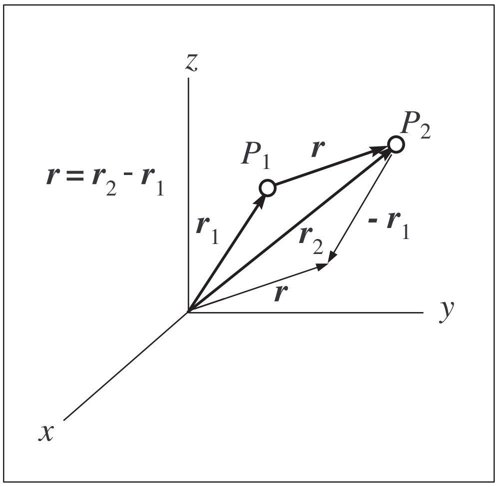

Figure 4.1: Attraction of a point mass m, located in point P_1, on P_2.

$$= -G \frac{m}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{3/2}} \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} .$$

Superposition—discrete

Gravitational formulae were derived for single point masses so far. One important property of gravitation is the so-called superposition principle. It says that the gravitational attraction of a system of masses can be achieved simply by adding the attractions of single masses. The m_i are the single masses and the r_i are the distances between mass points and the evaluation point.

$$\boldsymbol{a} = -G \sum_i \frac{m_i}{r_i^3} \boldsymbol{r}_i . \tag{4.3}$$

Superposition—continuous

Real world mass configurations can be thought of as systems of infinitely many and infinitely close point masses. The discrete formulation will become a continuous one.

$$N \to \infty$$
$$\sum_i \to \iiint_\Omega$$
$$m_i \to \mathrm{d}m$$

The body Ω consists of mass elements $\mathrm{d}m$, that are the infinitesimal masses of infinitesimal cubes $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ with local density $\rho(x, y, z)$:

$$\mathrm{d}m(x, y, z) = \rho(x, y, z)\, \mathrm{d}x\mathrm{d}y\mathrm{d}z . \tag{4.4}$$

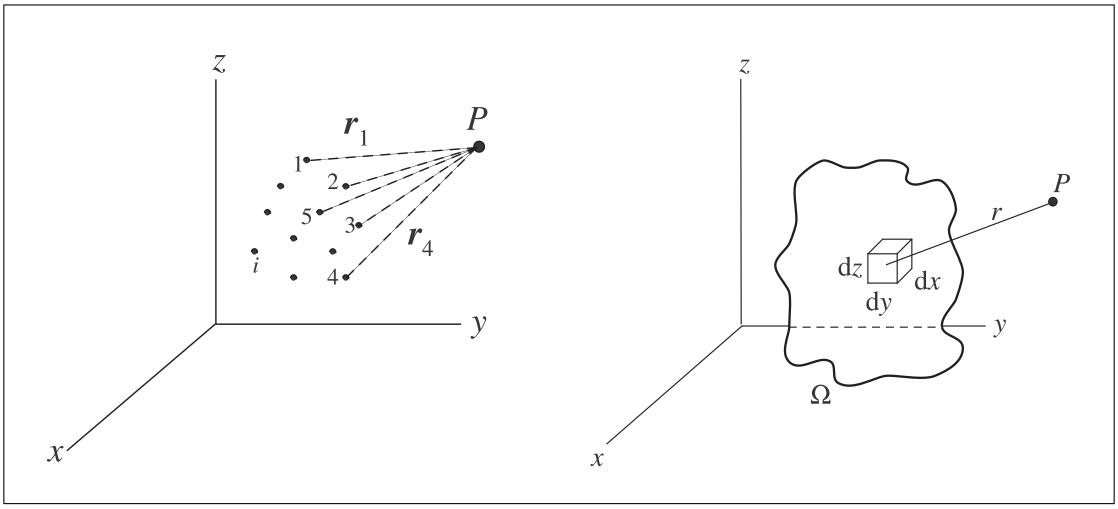

Figure 4.2.: Superposition for discrete (left) and continuous (right) mass distributions.

Integrating over all mass elements in Ω—the continuous equivalent of superposition—gives the attraction generated by Ω:

$$\boldsymbol{a} = -G \iiint\limits_{\Omega} \frac{\rho(x, y, z)}{r^3} \boldsymbol{r} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \,, \tag{4.5}$$

with r the distance between computation point P and mass element dm. The attraction (4.5) can in principle be determined using volume integrals if the density distribution within the body Ω is known. However, we can obviously not apply these integrals to the real Earth. The Earth's internal density distribution is insufficiently known. For that reason we will make use of potential theory to turn the volume integrals into surface integrals in a later chapter.

4.2. Rotation

kinematics Gravity related measurements take generally place on non-static platforms: sea-gravimetry, airborne gravimetry, satellite gravity gradiometry, inertial navigation. Even measurements on a fixed point on Earth belong to this category because of the Earth's rotation. Accelerated motion of the reference frame induces inertial accelerations, which must be taken into account in physical geodesy. The rotation of the Earth causes a centrifugal acceleration which is combined with the gravitational attraction into a new quantity: *gravity*. Other inertial accelerations are usually accounted for by correcting the gravity related measurements, e.g. the Eötvös correction. For these and other purposes we will start this chapter by investigating velocity and acceleration in a rotating frame.

Let us consider the situation of motion in a *rotating* reference frame and let us associate this rotating frame with the Earth-fixed frame. The following discussion on velocities and accelerations would be valid for any rotating frame, though.

Inertial coordinates, velocities and accelerations will be denoted with the index i. Earth-fixed quantities get the index e. Now suppose that a *time-dependent* rotation matrix $R = R(\alpha(t))$, applied to the inertial vector $\boldsymbol{r}_i$, results in the Earth-fixed vector $\boldsymbol{r}_e$. We would be interested in velocities and accelerations in the rotating frame. The time derivations must be performed in the inertial frame, though.

From $R\boldsymbol{r}_i = \boldsymbol{r}_e$ we get:

$$\boldsymbol{r}_i = R^{\mathsf{T}}\boldsymbol{r}_e \tag{4.6a}$$

$$\Downarrow \text{ time derivative}$$

$$\dot{\boldsymbol{r}}_i = R^{\mathsf{T}}\dot{\boldsymbol{r}}_e + \dot{R}^{\mathsf{T}}\boldsymbol{r}_e \tag{4.6b}$$

$$\Downarrow \text{ multiply by } R$$

$$\begin{aligned} R\dot{\boldsymbol{r}}_i &= \dot{\boldsymbol{r}}_e + R\dot{R}^{\mathsf{T}}\boldsymbol{r}_e \\ &= \dot{\boldsymbol{r}}_e + \Omega\boldsymbol{r}_e \end{aligned} \tag{4.6c}$$

The matrix $\Omega = R\dot{R}^{\mathsf{T}}$ is called *Cartan*[4] *matrix*. It describes the rotation rate, as can be seen from the following simple 2D example with $\alpha(t) = \omega t$:

$$R = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}$$

$$\Rightarrow \ \Omega = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \omega \begin{pmatrix} -\sin\omega t & -\cos\omega t \\ \cos\omega t & -\sin\omega t \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

It is useful to introduce Ω. In the next time differentiation step we can now distinguish between time dependent rotation matrices and time variable rotation rate. Let's pick

[4] Élie Joseph Cartan (1869–1951), French mathematician.

up the previous derivation again:

$$
\begin{aligned}
&\Downarrow \text{ multiply by } R^{\mathsf{T}} \\
\dot{\boldsymbol{r}}_i &= R^{\mathsf{T}}\dot{\boldsymbol{r}}_e + R^{\mathsf{T}}\Omega\boldsymbol{r}_e && (4.6\text{d}) \\
&\Downarrow \text{ time derivative} \\
\ddot{\boldsymbol{r}}_i &= R^{\mathsf{T}}\ddot{\boldsymbol{r}}_e + \dot{R}^{\mathsf{T}}\dot{\boldsymbol{r}}_e + \dot{R}^{\mathsf{T}}\Omega\boldsymbol{r}_e + R^{\mathsf{T}}\dot{\Omega}\boldsymbol{r}_e + R^{\mathsf{T}}\Omega\dot{\boldsymbol{r}}_e \\
&= R^{\mathsf{T}}\ddot{\boldsymbol{r}}_e + 2\dot{R}^{\mathsf{T}}\dot{\boldsymbol{r}}_e + \dot{R}^{\mathsf{T}}\Omega\boldsymbol{r}_e + R^{\mathsf{T}}\dot{\Omega}\boldsymbol{r}_e && (4.6\text{e}) \\
&\Downarrow \text{ multiply by } R \\
R\ddot{\boldsymbol{r}}_i &= \ddot{\boldsymbol{r}}_e + 2\Omega\dot{\boldsymbol{r}}_e + \Omega\Omega\boldsymbol{r}_e + \dot{\Omega}\boldsymbol{r}_e \\
&\Downarrow \text{ or the other way around} \\
\ddot{\boldsymbol{r}}_e &= R\ddot{\boldsymbol{r}}_i - 2\Omega\dot{\boldsymbol{r}}_e - \Omega\Omega\boldsymbol{r}_e - \dot{\Omega}\boldsymbol{r}_e && (4.6\text{f})
\end{aligned}
$$

This equation tells us that acceleration in the rotating e-frame equals acceleration in the inertial i-frame—in the proper orientation, though—when 3 more terms are added. The additional terms are called *inertial* accelerations Analyzing (4.6f) we can distinguish the four terms at the right hand side:

- $R\ddot{\boldsymbol{r}}_i$ is the inertial acceleration vector, expressed in the orientation of the rotating frame.
- $2\Omega\dot{\boldsymbol{r}}_e$ is the so-called *Coriolis*[5] acceleration, which is due to motion in the rotating frame.
- $\Omega\Omega\boldsymbol{r}_e$ is the *centrifugal* acceleration, determined by the position in the rotating frame.
- $\dot{\Omega}\boldsymbol{r}_e$ is sometimes referred to as *Euler*[6] acceleration or inertial acceleration of rotation. It is due to a non-constant rotation rate.

Remark 4.3 *Equation (4.6f) can be generalized to moving frames with time-variable origin. If the linear acceleration of the e-frame's origin is expressed in the i-frame with $\ddot{\boldsymbol{b}}_i$, the only change to be made to (4.6f) is $R\ddot{\boldsymbol{r}}_i \rightarrow R(\ddot{\boldsymbol{r}}_i - \ddot{\boldsymbol{b}}_i)$.*

Properties of the Cartan matrix $\boldsymbol{\Omega}$. Cartan matrices are skew-symmetric, i.e. $\Omega^{\mathsf{T}} = -\Omega$. This can be seen in the simple 2D example above already. But it also follows from the orthogonality of rotation matrices:

$$RR^{\mathsf{T}} = I \implies \frac{\mathrm{d}}{\mathrm{d}t}(RR^{\mathsf{T}}) = \underbrace{\dot{R}R^{\mathsf{T}}}_{\Omega^{\mathsf{T}}} + \underbrace{R\dot{R}^{\mathsf{T}}}_{\Omega} = 0 \implies \Omega^{\mathsf{T}} = -\Omega\,. \tag{4.7}$$

[5] Gaspard Gustave de Coriolis (1792–1843).
[6] Leonhard Euler (1707–1783).

A second interesting property is the fact that multiplication of a vector with the Cartan matrix equals the cross product of the vector with a corresponding rotation vector:

$$\Omega \boldsymbol{r} = \boldsymbol{\omega} \times \boldsymbol{r} \tag{4.8}$$

This property becomes clear from writing out the 3 Cartan matrices, corresponding to the three independent rotation matrices:

$$\left.\begin{array}{l} R_1(\omega_1 t) \;\Rightarrow\; \Omega_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega_1 \\ 0 & \omega_1 & 0 \end{pmatrix} \\ R_2(\omega_2 t) \;\Rightarrow\; \Omega_2 = \begin{pmatrix} 0 & 0 & \omega_2 \\ 0 & 0 & 0 \\ -\omega_2 & 0 & 0 \end{pmatrix} \\ R_3(\omega_3 t) \;\Rightarrow\; \Omega_3 = \begin{pmatrix} 0 & -\omega_3 & 0 \\ \omega_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}\right\} \overset{\text{general}}{\Longrightarrow} \quad \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} . \tag{4.9}$$

Indeed, when a general rotation vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^{\mathsf{T}}$ is defined, we see that:

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

The skew-symmetry (4.7) of Ω is related to the fact $\boldsymbol{\omega} \times \boldsymbol{r} = -\boldsymbol{r} \times \boldsymbol{\omega}$.

Exercise 4.1 *Convince yourself that the above Cartan matrices Ω_i are correct, by doing the derivation yourself. Also verify (4.8) by writing out* LHS *and* RHS.

Using property (4.8), the velocity (4.6c) and acceleration (4.6f) may be recast into the perhaps more familiar form:

$$\dot{\boldsymbol{r}}_e = R\dot{\boldsymbol{r}}_i - \boldsymbol{\omega} \times \boldsymbol{r}_e \tag{4.10a}$$

$$\ddot{\boldsymbol{r}}_e = R\ddot{\boldsymbol{r}}_i - 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_e - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_e) - \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_e \tag{4.10b}$$

Inertial acceleration due to Earth rotation

Neglecting precession, nutation and polar motion, the transformation from inertial to Earth-fixed frame is given by:

$$\boldsymbol{r}_e = R_3(\text{GAST})\boldsymbol{r}_i \overset{\text{or}}{\to} \boldsymbol{r}_e = R_3(\omega t)\boldsymbol{r}_i \, . \tag{4.11}$$

The latter is allowed here, since we are only interested in the acceleration effects, due to the rotation. We are not interested in the rotation of position vectors. With great precision, one can say that the Earth's rotation rate is constant: $\dot{\omega} = 0$ The corresponding Cartan matrix and its time derivative read:

$$\Omega = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \dot{\Omega} = 0\,.$$

The three inertial accelerations, due to the rotation of the Earth, become:

$$\text{Coriolis:} \qquad -2\Omega\dot{\boldsymbol{r}}_e = 2\omega \begin{pmatrix} \dot{y}_e \\ -\dot{x}_e \\ 0 \end{pmatrix} \tag{4.12a}$$

$$\text{centrifugal:} \quad -\Omega\Omega\boldsymbol{r}_e = \omega^2 \begin{pmatrix} x_e \\ y_e \\ 0 \end{pmatrix} \tag{4.12b}$$

$$\text{Euler:} \qquad -\dot{\Omega}\boldsymbol{r}_e = \boldsymbol{0} \tag{4.12c}$$

The Coriolis acceleration is perpendicular to both the velocity vector and the Earth's rotation axis. It will be discussed further in **??**. The centrifugal acceleration is perpendicular to the rotation axis and is parallel to the equator plane, cf. fig. 4.4.

Exercise 4.2 *Determine the direction and the magnitude of the Coriolis acceleration if you are driving from Calgary to Banff with* 100 km/h.

Exercise 4.3 *How large is the centrifugal acceleration in Calgary? On the equator? At the North Pole? And in which direction?*

4.3. Gravity

Suppose we are doing gravitational measurements at a fixed location on the surface of the Earth. So $\dot{\boldsymbol{r}}_e = 0$ and the Coriolis acceleration in (4.12) vanishes. The only remaining term is the centrifugal acceleration $\boldsymbol{a}_\mathrm{c}$, specified in the e-frame by: $\boldsymbol{a}_\mathrm{c} = \omega^2(x_e,\ y_e,\ 0)^\mathsf{T}$. Since this acceleration is always present, it is usually added to the gravitational attraction. The sum is called *gravity*:

gravity = gravitational attraction + centrifugal acceleration

$$\boldsymbol{g} = \boldsymbol{a} + \boldsymbol{a}_\mathrm{c}\ .$$

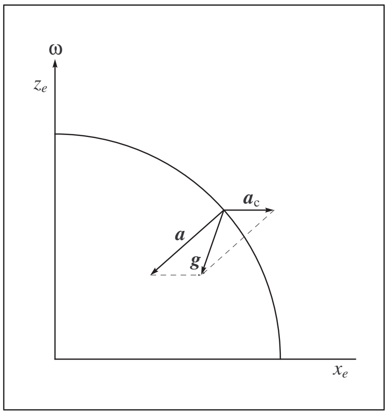

Figure 4.3: Gravity is the sum of gravitational attraction and centrifugal acceleration. Note that $\boldsymbol{a}_c$ is hugely exaggerated. The centrifugal acceleration vector is about 3 orders of magnitude smaller than the gravitational attraction.

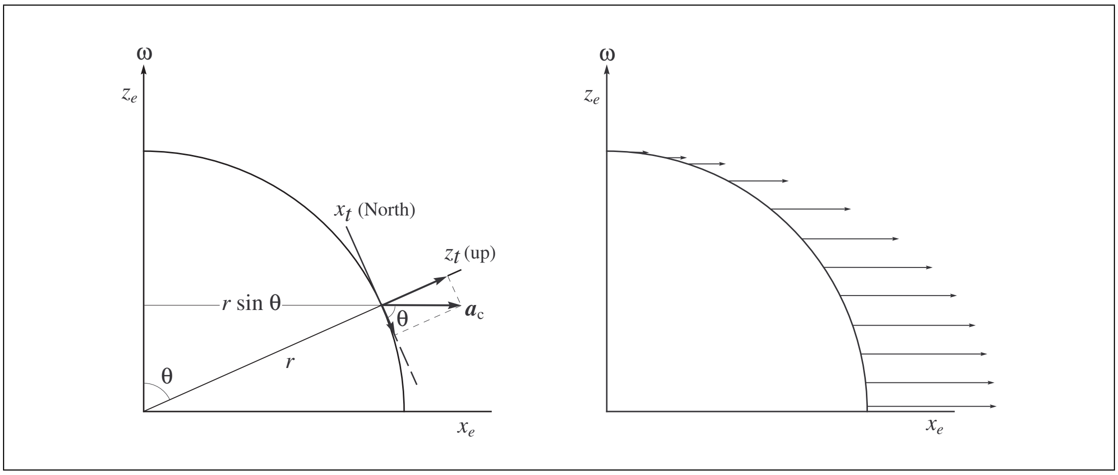

Figure 4.4.: Centrifugal acceleration in Earth-fixed and in topocentric frames.

Centrifugal acceleration in the local frame. Since geodetic observations are usually made in a local frame, it makes sense to express the centrifugal acceleration in the following topocentric frame (t-frame):

- x-axis tangent to the local meridian, pointing North,
- y-axis tangent to shperical latitude circle, pointing East, and
- z-axis complementary in left-handed sense, point up.

Note that this is is a left-handed frame. Since it is defined on a sphere, the t-frame can be considered as a spherical approximation of the local astronomic g-frame. Vectors in the Earth-fixed frame are transformed into this frame by the sequence:

$$\boldsymbol{r}_t = P_1 R_2(\theta) R_3(\lambda) \boldsymbol{r}_e = \begin{pmatrix} -\cos\theta\cos\lambda & -\cos\theta\sin\lambda & \sin\theta \\ -\sin\lambda & \cos\lambda & 0 \\ \sin\theta\cos\lambda & \sin\theta\sin\lambda & \cos\theta \end{pmatrix} \boldsymbol{r}_e \,, \tag{4.13}$$

in which λ is the longitude and θ the co-latitude. The mirroring matrix $P_1 = \mathrm{diag}(-1, 1, 1)$ is required to go from a right-handed into a left-handed frame. Note that we did not include a translation vector to go from geocenter to topocenter. We are only interested in directions here. Applying the transformation now to the centrifugal acceleration vector in the e-frame yields:

$$\boldsymbol{a}_{\mathrm{c},t} = P_1 R_2(\theta) R_3(\lambda) r\omega^2 \begin{pmatrix} \sin\theta\cos\lambda \\ \sin\theta\sin\lambda \\ 0 \end{pmatrix} = r\omega^2 \begin{pmatrix} -\cos\theta\sin\theta \\ 0 \\ \sin^2\theta \end{pmatrix} = r\omega^2 \sin\theta \begin{pmatrix} -\cos\theta \\ 0 \\ \sin\theta \end{pmatrix} . \tag{4.14}$$

The centrifugal acceleration in the local frame shows no East-West component. On the Northern hemisphere the centrifugal acceleration has a South pointing component. For gravity purposes, the vertical component $r\omega^2 \sin^2\theta$ is the most important. It is always pointing up (thus reducing the gravitational attraction). It reaches its maximum at the equator and is zero at the poles.

A. The Greek alphabet

α	A	alpha
β	B	beta
γ	Γ	gamma
δ	Δ	delta
ϵ, ε	E	epsilon
ζ	Z	zeta
η	H	eta
θ, ϑ	Θ	theta
ι	I	iota
κ	K	kappa
λ	Λ	lambda
μ	M	mu
ν	N	nu
ξ	Ξ	ksi
o	O	o*micron*
π, ϖ	Π	pi
ρ, ϱ	P	rho
σ, ς	Σ	sigma
τ	T	tau
υ	Υ	upsilon
ϕ, φ	Φ	phi
χ	X	chi
ψ	Ψ	psi
ω	Ω	o*mega*